A class of cardinal lacunary interpolation problems by spline functions

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Let $n$, $r$ be positive integers with $n \geq 2r - 1$ and let $p$, $q$, $k$ be non-negative integers such that

$$r = p + q, \quad 1 \leq p \leq r, \quad 0 \leq q \leq r - 1, \quad p \leq k \leq n - r - q + 1.$$  \tag{1}

For integers $s$, $t$ such that $0 \leq t \leq r$ and $0 \leq s \leq n - r + 1$, let $S_{n,r}^{s,t}$ denote the class of cardinal spline functions $S(x)$ satisfying the following conditions:

1. $S(x)$ is a polynomial of degree $n$ in each of the intervals $[v, v + 1]$ for $v = 0, \pm 1, \pm 2, \ldots$.
2. $S(x) \in C^{r-1}(-\infty, \infty)$ and $S^{(p)}(v +) = S^{(p)}(v -) (\rho = s + t, s + t + 1, \ldots, n - r + 1) \forall$ integers $\nu$. \tag{3}

We shall consider the following

PROBLEM. Given positive integers $p$, $q$, $k$ satisfying (1) and $r$ bi-infinite sequences of numbers

$$y^{(i)} = (y_{v}^{(i)}) \quad (i = 0, 1, \ldots, p - 1, k, \ldots, k + q - 1)$$  \tag{4}

satisfying

$$y_{v}^{(i)} = 0(|v|^{\gamma}) \quad (i = 0, 1, \ldots, p - 1, k, \ldots, k + q - 1)$$  \tag{5}

for some $\gamma > 0$, to find a function $S(x) \in S_{n,r}^{s,t}$ such that

$$S^{(i)}(\nu) = y_{\nu}^{(i)} \quad (i = 0, 1, \ldots, p - 1, k, \ldots, k + q - 1)$$  \tag{6}

$\forall$ integers $\nu$.

We shall call the above problem Cardinal Lacunary Interpolation Problem (C.L.I.P.) of type $(0, p; k, q)$ in $S_{n,r}^{s,t}$. If $n = 2m - 1$, this problem has a unique solution provided one of the following conditions is satisfied:

$$t = 0, \quad k = 2m - r - q, \quad m \geq r + 1,$$

or $t$ and $q$ are both even and $0 \leq t < k + q < s \leq 2m - r$, \tag{8}
t is even, \( q = 1 \) and \( 0 \leq t < r - 1 < k \leq s - 1 \leq 2m - r - 1 \) with \( (k - r) \) odd, \( (9) \)

\( t = 1, \quad q = 1, \quad s = 2m - r - 1 \) and \( 0 < r - 1 < k \leq 2m - r - 2 \) with \( (k - r) \) even. \( (10) \)

More precisely we have

**THEOREM 1.** Let \( p, q \) satisfy (1) and further suppose that \( p, q, k, s, t \) satisfy one of the conditions (7)-(10). Then given \( r \) bi-infinite sequences (4) satisfying (5) the C.L.I.P. of type \( (0, p; k, q) \) has a unique solution \( S(x) \in \Sigma^{\pm}_{2m-r} \) such that

\[
S^{(\rho)}(x) = 0(|x|^\gamma) \quad (\rho = 0, 1, \ldots, p - 1, k, \ldots, k + q - 1) \text{ as } |x| \rightarrow \infty. \quad (11)
\]

The proof of Theorem 1 follows the same approach as Lipow and Schoenberg [2]. For the C.L.I.P. the main difficulty of this approach is to show that the polynomials

\[
\Pi_{n,t}^{s,t}(0, p; k, q; \lambda) = P_{\lambda}\left(\begin{array}{c} r - t, \ldots, n - s - t, n - s + 1, \ldots, n \\ 0, \ldots, n - k - q, n - k + 1, \ldots, n - p \end{array}\right)
\]

has real simple zeros. Here we use the notation

\[
P_{\lambda}(i_0, i_1, \ldots, i_n)
\]

for the determinant of the submatrix obtained by deleting from

\[
P - \lambda I = \begin{vmatrix} i_i \\ j_j \end{vmatrix} - \lambda \delta_{ij} \quad (i, j = 0, 1, \ldots)
\]

all the rows and columns except those numbered \( i_0, i_1, \ldots, i_n \) and \( j_0, j_1, \ldots, j_n \) respectively.

**THEOREM 2.** Let \( p, q, k \) be integers satisfying (1), \( s, t \) be integers with \( 0 \leq t < r < k + q \leq s \leq n - r \) and suppose that \( q, t \) are both even. Then the zeros of the polynomials \( \Pi_{n,r}^{s,t}(0, p; k, q; \lambda), \Pi_{n-1,r}^{s,t}(0, p; k, q; \lambda) \) and \( \Pi_{n-1,r}^{s-1,t}(0, p; k, q; \lambda) \) are real, simple and of sign \((-1)^p\). Furthermore the zeros of \( \Pi_{n,r}^{s,t}(0, p; k, q; \lambda) \) interlace with the zeros of \( \Pi_{n-1,r}^{s-1,t}(0, p; k, q; \lambda) \) (respectively \( \Pi_{n-1,r}^{s-1,t}(0, p; k, q; \lambda) \)).

When \( q = t = 0 \), Theorem 2 reduces to the main result of Lipow and Schoenberg in [2], and shows further the interlacing property of the zeros. Our proof of