Generalized-Finslerian connection coefficients for the static spherically symmetric space

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Summary. The metric torsionless connection coefficients are found in an explicit way in the case of static spherically symmetric spaces defined in a generalized-Finslerian way. The connection coefficients are determined in terms of the metric tensor and its first derivatives.

1. Introduction

Suppose we are given an N-dimensional differentiable manifold \( M \) together with its local coordinates \( x^i \). Let us denote by \( y^i \) the components of a tangent vector (with respect to the natural frame) supported by \( x^i \), and by \( TM \) the tangent bundle of \( M \) without the zero section, so that the pair \( (x^i, y^i), \sum_{i=1}^{N} (y^i)^2 > 0 \) is an element of \( TM \). Then we can consider on \( TM \) a symmetric nondegenerate tensor of type \( (0, 2) \), whose local components are denoted by \( a_{ij}(x, y) \). We shall assume that \( a_{ij} \) is positively homogeneous of degree zero in \( y^i \) and differentiable with respect to each of its \( 2N \) arguments (except when all the \( y^i \) vanish). The indices \( i, j, \ldots \) will range from 0 to \( N - 1 \).

Let us put

\[
C_{mni} = \frac{1}{2} \frac{\partial a_{mn}}{\partial y^i}.
\]

If

\[
C_{mni} y^m = 0
\]


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then \( a_{ij} \) is a Finslerian metric tensor. However, the geometric limits assigned by the theory of Finsler spaces (described in [1, 4]) can effectively be extended by dropping the condition (1.2). The corresponding general differential-geometric approach was formulated in [5, 6] in detail.

On introducing connection coefficients \( L_{im}^j(x, y) \) on \( TM \), we can follow [6] and Section 7 of [5] to stipulate that the metric condition

\[
\partial_i a_{mn} - 2y^k L_{kl}^i C_{mnj} - L_{mi}^k a_{kn} - L_{mi}^k a_{mk} = 0
\]

(1.3)

holds, where \( \partial_i = \partial / \partial x^i \), in which case the connection coefficients \( L_{im}^j \) are called metric relative to the tensor \( a_{ij}(x, y) \). In the sequel, we shall assume that \( L_{im}^j \) are positively homogeneous of degree zero in \( y^i \) and symmetric with respect to their subscripts. From (1.3) it follows directly that

\[
L_{jk}^i = r_{jk}^i - a^{im}(L_{jm}^m C_{kn} + L_{jm}^m C_{km} - L_{jm}^m C_{km}),
\]

(1.4)

which in turn entails

\[
L_k^i = Q_k^i - a^{im}(L_{km}^m C_{jm} - L_{km}^m C_{jm}),
\]

(1.5)

with

\[
Q_k^i = r_k^i - a^{im}L_{km}^m C_{kn},
\]

(1.6)

where

\[
L_{kj}^i = L_{kj}^i, \quad L^i = L_{kj}^i y^k, \quad r_k^i = r_{kj}^i y^j,
\]

(1.7)

and

\[
r_{kj}^i = \frac{1}{2} a^{im}(\partial_k a_{jn} + \partial_j a_{kn} - \partial_n a_{kj})
\]

(1.8)

are the associated Christoffel symbols. The tensor \( a^{ij} \) is reciprocal to \( a_{ij} \), so that \( a^{ij} a_{jn} = \delta_i^j \).

The equations (1.4) cannot be resolved for \( L_{jk}^i \) in the general case. Necessary and sufficient condition which the coefficients \( L_{jk}^i \) must satisfy to solve the equations (1.4) were established in Section 7 of [5] (not even supposing that the fields \( a_{ij} \) and \( L_{jk}^i \) are homogeneous and the coefficients \( L_{jk}^i \) are symmetric). In Section 2, we consider an interesting particular metric tensor proposed earlier in [2]. In the