The translation equation on certain \( n \)-groups

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Summary. In this note the characterization of all solutions of the equation

\[
\Phi(\Phi(\ldots \Phi(x, x_1), x_2), \ldots, x_{n-1}), x_n) = \Phi(x, x_1 \cdot x_2 \cdot \ldots \cdot x_n),
\]

where \( n \geq 3 \), \( \Phi: \Gamma \times G \to \Gamma \)

and \((G, \cdot)\) is a group, is given. This equation represents the translation equation on particular \( n \)-groups which are obtained from a binary group \((G, \cdot)\) by definition of an \( n \)-group operation as follows: \([x_1, \ldots, x_2] := x_1 \cdot x_2 \cdot \ldots \cdot x_n\). We will show that, in order to characterize all solutions of the above mentioned equation, it is necessary and sufficient to describe all solutions of the translation equation \( F(F(x, x), y) = F(x, x \cdot y) \) fulfilling certain conditions.

**Introduction**

Let \( n \geq 3 \) be a natural number and let \( \Gamma \) be an arbitrary set. By \((G, [\ldots])\) we denote an arbitrary \( n \)-adic group (see [4]).

The translation equation on the \( n \)-group \((G, [\ldots])\) is the following equation:

\[
\Phi(\Phi(\ldots \Phi(x, x_1), x_2), \ldots, x_{n-1}), x_n) = \Phi(x, [x_1 x_2 \ldots x_n]),
\]

where \( \Phi: \Gamma \times G \to \Gamma \). (1)

By a result of Hosszú [1] the \( n \)-group operation in \( G \) can be expressed by using a binary group operation on \( G \) and some automorphism of it in the following way:

\[ [x_1 x_2 \ldots x_n] = x_1 \cdot \mu(x_2) \cdot \mu^2(x_3) \cdot \ldots \cdot \mu^{n-1}(x_n) \cdot a, \]

where "·" is a binary group operation on \( G \) and \( \mu \) is an automorphism of \((G, \cdot)\) and \( a \in G, \mu(a) = a \) and \( \mu^{n-1}(x) = a \cdot x \cdot a^{-1} \) (by \( \mu^v \) we denote the \( v \)th iteration).


Therefore, the equation (1) can be expressed in the form

\[ \Phi(\Phi(\ldots \Phi(\Phi(x, x_1), x_2), \ldots), x_{n-1}), x_n) = \Phi(x, x_1 \cdot \mu(x_2) \cdot \mu^2(x_3) \cdot \ldots \cdot \mu^{n-1}(x_n) \cdot a). \]  
(1')

If we set \( a = e \) in (1'), where \( e \) is the unit element of \((G, \cdot)\) and \( \mu \) is the identity on \( G \), we obtain the equation

\[ \Phi(\Phi(\ldots \Phi(\Phi(x, x_1), x_2), \ldots), x_{n-1}), x_n) = \Phi(x, x_1 \cdot x_2 \cdot \ldots \cdot x_n), \]  
(2)

where \( \Phi: \Gamma \times G \rightarrow \Gamma \) and \((G, \cdot, e)\) is a group.

Therefore, the equation (2) represents the translation equation on \( n \)-groups which are obtained from a binary group \((G, \cdot)\) by defining the \( n \)-group operation as follows:

\[ [x_1 x_2 \ldots x_n] = x_1 \cdot x_2 \cdot \ldots \cdot x_n. \]

Let us observe that every solution \( F: \Gamma \times G \rightarrow \Gamma \) of the translation equation

\[ F(F(\alpha, x), y) = F(\alpha, x \cdot y) \]  
(3)

satisfies the equation (2) (for every \( n \geq 3 \)). The converse is not true.

**EXAMPLE 1.** Let \( n = 3 \), \( \Gamma = \mathbb{R} \), where \( \mathbb{R} \) denotes the set of real numbers. Let \( (G, \cdot, e) = ((\mathbb{R} \setminus \{0\}, \cdot, 1) \) and \( \Phi(x, x) = -x \cdot x \). It is easy to check that \( \Phi \) satisfies equation (2) and \( \Phi \) does not satisfy equation (3).

**EXAMPLE 2.** Let \( n = 4 \), \( \Gamma = \{\gamma, \beta_0, \beta_1, \beta_2\} \). Let \((G, \cdot, e)\) be an arbitrary group. Define \( \Phi: \Gamma \times G \rightarrow \Gamma \) by the formula

\[ \Phi(\alpha, x) := \begin{cases} 
\beta_0 & \text{for } \alpha = \gamma, \\
\beta_{(i+1)} & \text{for } \alpha = \beta_i, \ i \in \{0, 1, 2\},
\end{cases} \]

where \( (i + 1)_3 \) denotes the remainder when \( i + 1 \) is divided by 3. Similarly as in Example 1, the function \( \Phi \) is a solution of equation (2) and \( \Phi \) does not satisfy the translation equation (3).

The general solution \( F: \Gamma \times G \rightarrow \Gamma \) of the translation equation (3) is given in the paper [5] by the following