SCATTERING CHARACTERISTICS OF A PROBLEM OF DIFFRACTION BY A WEDGE AND BY A SCREEN

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In the case of the scattering problem on a wedge and on a screen, for a certain class of boundary conditions, one constructs explicitly the wave operators and one establishes their completeness. It is shown that a modified scattering matrix (including additionally the reflection operator) is a unitary operator with a pure point spectrum. In the case of a screen, the standard S-matrix is unitary. For Dirichlet (Neumann) boundary conditions, the S-matrix is reduced explicitly to a diagonal form. The spectrum of the S-matrix is simple, absolutely continuous, filling the lower (upper) semicircumference.

1. Scattering by noncompact obstacles is associated with the appearance of effects not present for the case of compact obstacles. Since a general theory of such problems has not been constructed, we analyze these effects here using the example of the exactly solvable problems on a wedge and on a screen. We show the existence of complete wave operators (with respect to the Laplace operator $-\Delta$ in a free space). In the case of a wedge the formal analogue of the usual definition of the scattering matrix reduces to a nonunitary operator. However, a certain modification of the definition gives a unitary operator, the matrix $\sigma$. The operator $\sigma$ has a pure point spectrum. In the case of a screen, the s-matrix is unitary. For Dirichlet ($\mathcal{D}$) and Neumann ($\mathcal{N}$) boundary conditions we calculate explicitly the kernel of the s-matrix. We give a representation in which the s-matrix is diagonal. For it one has an absolutely continuous spectrum.

2. Preliminary Information. For nonzero vectors $x, p$ of the real Euclidean space $\mathbb{R}^2$ we set $r = |x|$, $k = |p|$ and let $v = r^{-1}x$, $\omega = k^{-1}p$ points of the unit circumference in $\mathbb{R}^2$, which will be identified with their corresponding polar angles. Let $\mathcal{R} = \mathcal{R}_\alpha$ be a domain in $\mathbb{R}^2$, being the exterior of a wedge, bounded by the rays $v = 0$, $v = \alpha$ and having opening $2\pi - \alpha$. For $\alpha = 2\pi$ we obtain a semi-infinite screen. Let $\gamma_\alpha = \mathcal{L}(\mathcal{R}_\alpha)$, $\gamma = \mathcal{L}(\mathbb{R}^2)$, $\alpha = \mathcal{L}(0, \alpha)$, $\mathcal{N}_\gamma = \mathcal{L}(\mathbb{R}, \gamma d\nu)$. In the Hilbert space $\mathcal{H}$ we consider the self-adjoint boundary value problem for the equation $\partial^2 u/\partial \varphi^2 - \beta_0 u = 0$, $\beta_0 > 0$, $\beta_0 < \infty$. The numbers $\mu_n$, $\mu_n > 0$ and the functions $Q_n$ are its eigenvalues and normalized eigenfunctions, respectively. In $\gamma_\alpha$ we consider the self-adjoint operator $H$, corresponding to the operator $-\Delta$ under the conditions $[\partial u/\partial \varphi + \beta_1 u]_{\varphi = \alpha} = [\partial u/\partial \varphi + \beta_1 u]_{\varphi = -\alpha} = 0$. The operator $H_0$ is generated by the differential expression $-\Delta$ in $\gamma_{\gamma_0}$. The symbol $\gamma$ denotes the mapping $u(x) \mapsto u(-x)$. The function (see [1])

$$u(x,p) = 2\pi \sum_{n=0}^{\infty} Q_n(x) \Phi_n(x) \Phi_n(p) \exp(-i\pi\mu_n/2)$$

is the solution of the problem on the diffraction of a plane wave with the impulse $(-\mathbf{p})$ on the wedge. Let $F$ be the Fourier operator in $\gamma_\alpha$. In $\gamma$ we introduce the "generalized Fourier operators": $\Phi_\alpha = \Phi x$, $\Phi^* \Phi \gamma = 2\pi \int u(x,p) f(x) dx$, $2\pi \gamma = \int u(x,p) g(p) dp$. The symbol $\gamma$ denotes the mapping $u(x) \mapsto u(-x)$. The function (see [1])

$$2\pi \gamma(p) = \int u(x,p) f(x) dx$$

The mappings (2) are unitary in \( \psi_{\theta} \). Let \( X_{n} \) be the Hankel transform, unitary in \( \mathcal{H} \),

\[
(X_{n} f_{n})(\nu) = \tilde{f}_{n}(\nu) = \int_{\mathbb{R}} \mathcal{J}_{\nu_{n}}(\kappa
nu) \psi_{n}(\kappa) \kappa \, \text{d}\kappa.
\]

The operators \( X: \{ f_{n} \} \rightarrow \{ \tilde{f}_{n} \} \), \( M: \{ f_{n} \} \rightarrow \{ \exp(-i\pi \mu_{n}/\nu) \tilde{f}_{n} \} \) are unitary in \( \mathcal{H} \). Assume that the operator \( \mathcal{D}: \psi_{\theta} \rightarrow \mathcal{L}(\mathcal{H}) \) maps the function \( \psi \in \mathcal{H} \) into the sequence \( \{ \tilde{f}_{n} \} \) of its Fourier coefficients with respect to the system \( \{ \psi_{n} \} \). Then \( \Phi_{\theta} = \mathcal{D} \mathcal{X} \mathcal{M} \).

3. Wave Operators. Let \( T: \psi_{\theta} \rightarrow \psi_{\theta} \) be the operator of continuation by zero on \( \mathbb{R}^{2} \); then \( T^{*}: \psi_{\theta} \rightarrow \psi_{\theta} \) is the operator of restriction to \( \Omega \). The symbol \( \Pi_{+}: \psi_{\theta} \rightarrow \psi_{\theta} \) denotes the multiplication operator by the indicator of the set \( \Omega \); \( \Pi_{=} = \mathcal{J}_{+} \mathcal{J}_{-} \), \( P_{=} = F^{*} \mathcal{P} \). We introduce the subspaces \( \psi_{\theta}^{\pm} = P_{=} \psi_{\theta} \); then \( \mathcal{J}_{=} = \psi_{\theta}^{\pm} \).

For the wave operators

\[
W_{\pm}(\mathcal{H}, \mathcal{H}) = \lim_{t \rightarrow \pm \infty} \exp(itH)T^{*}\exp(-itH_{0})
\]

simultaneously with the existence proof one establishes explicit formulas. From these formulas there follows also the completeness of \( W_{+} \).

**Theorem.** For the operators (3) we have the formulas

\[
W_{+} = \Phi_{+}^{*}B^{*}T_{+}^{*}F_{+}, \quad W_{-} = \Phi_{-}^{*}T_{-}^{*}F_{-},
\]

(4)

\[
B = \mathcal{D} \mathcal{M} \mathcal{D}.
\]

(5)

From (4) there follows that the operators \( W_{\pm} \) map unitarily \( \psi_{\theta}^{\pm} \) onto \( \psi_{\theta} \). Thus, the operators \( W_{\pm} \) are complete.

In the case of the screen \( \psi_{=} = \psi_{\theta} = \psi_{\theta} \), formulas (4) have a simpler form:

\[
W_{+} = \Phi_{+}^{*}B^{*}F, \quad W_{-} = \Phi_{-}^{*}F.
\]

(6)

4. Modified Scattering Matrix. In \( \psi_{\theta} \), the operators \( W_{\pm} \) are partially isometric operators of the space \( \psi_{\theta} \), their isometry domains being distinct. Therefore, the scattering operator \( S = W_{-}W_{+}^{*} \) is not unitary. Instead of \( S \) it is natural\(^{†}\) to consider the operator \( \Sigma = SF = W_{-}^{*}T_{+}^{*}F \), mapping \( \psi_{+} \) into itself. The unitary part of \( \Sigma \) is mapped into the impulse representation of the operator \( B \). The operator \( \Sigma \) commutes with the group \( \exp(-itH_{0}) \). This allows us to introduce the "sigma-matrix" (the analogue of the scattering matrix), i.e. the operator \( \sigma: \mathcal{A} \rightarrow \mathcal{A} \), defined by the equality \( \sigma g_{\kappa}(\kappa, \nu)(\omega) = (Bg)(\kappa \omega) \). The operator \( \sigma \) does not depend explicitly on "\( \kappa \)" which corresponds to the invariance of the problem relative to the homotheties: \( x \mapsto cx, \ c > 0 \). From (5) there follows that the operator \( \sigma \) is unitarily equivalent to the operator of multiplication by \( \exp(-i\pi \mu_{n}) \) in \( \mathbb{R}^{2} \). The spectrum of the operator \( \sigma \) is a pure point spectrum.

Under the conditions (9) we have \( \mu_{n} = \pi n / \alpha, \ n = 1, 2, \ldots \); under the conditions (N), \( \mu_{n} = \pi n / \alpha, \ n = 0, 1, 2, \ldots \). If \( \alpha / \pi \) is rational, then the spectrum \( \sigma \) consists of a finite number of eigenvalues of infinite multiplicity. In particular, for \( \alpha = 2\pi \) the spectrum consists of four points \( \pm 1, \pm i \). If \( \alpha = \pi / 2 \), \( z = 1, 2, \ldots \), then \( \sigma \) is the identity operator. For irrational \( \alpha / \pi \), the point spectrum of \( \sigma \) is dense on the unit circumference.

\(^{†}\)The allowed momenta of the incident waves are in the region \( (-\infty, 0) \), whereas for the momenta of free-propagating reflected waves the allowed region is \( \Omega \). Therefore, it is reasonable to introduce the identification \( \gamma \) between \( \psi_{\theta} \) and \( \psi_{\theta}^{\pm} \), taking into account this geometric picture. Such an identification does not necessarily exist for \( \alpha = \pm 2\pi \).