DISTRIBUTION OF INTEGRAL FUNCTIONALS OF A BROWNIAN MOTION PROCESS

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In this paper one considers methods which enable one to determine the distribution of certain functionals of a Brownian motion process. Among such functionals we have: the positive continuous additive functional of a Brownian motion, defined by the formula

$$A(t) = \int_{-\infty}^{\infty} \mathcal{L}(t, y) dF(y),$$

where $\mathcal{L}(t, y)$ is the Brownian local time process while $F(y)$ is a monotonically increasing right continuous function; the functional

$$B(t) = \int_{-\infty}^{\infty} f(y, \mathcal{L}(t, y)) dy,$$

where $f(y, x)$ is a continuous function; and the functional

$$C(t) = \int_{0}^{t} f(W(s), \mathcal{L}(s, y)) ds.$$

As an application of these methods one considers some concrete functionals such that $\mathbb{E}[\mathcal{L}(T, y)] = \mathbb{E}[\mathcal{L}(T, y)]$, where $T$ is an exponential random time, independent of $\mathcal{L}(t, y)$.

0. In this paper we consider methods which allow us to determine the distribution of certain functionals of the process of Brownian motion. The first class contains additive functionals of the Brownian motion. We denote by $w(t)$ the standard process of Brownian motion on a line. The limit (see [1])

$$\hat{\mathcal{L}}(t, y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{0}^{t} \mathcal{I}[y, y+\varepsilon](w(s)) ds,$$

where $\mathcal{I}$ is the indicator of the set $A$, exists with probability one and is called the Brownian local time at the point $y$ over the time $t$. In a certain sense, the Brownian local time is an elementary additive functional of the Brownian process. Thus, any positive continuous additive functional $A(t)$ of the Brownian motion can be represented (see [2]) in the form of the mixture

A(t) = \int_{-\infty}^{\infty} \hat{f}(t, y) dF(y) \quad a.s. \quad (0.1)

where \( F(y) \) is a monotonically increasing right-continuous function, defined on \( \mathbb{R}^+ \). The question of the determination of the distributions of such functionals is considered in Sec. 2 of the present paper.

In the particular case when the function \( F(y) \) has density \( f(y) \), the functional \( A(t) \) can be transformed into the form

\[
A(t) = \int_{-\infty}^{\infty} \hat{f}(t, y) f(y) dy = \int_{-\infty}^{\infty} f(w(s)) ds. \quad (0.2)
\]

The equations for the characteristic functions of such functionals have been obtained for the first time by Kac [3, 4]. In Sec. 1 we give the results of Kac and we give a new derivation of the equations for the characteristic functions of the functionals (0.2). In addition, based on the equations for the characteristic functions of functionals of the form (0.2) for homogeneous Markov processes, obtained by E. B. Dynkin, we give the equations for the determination of the characteristic functions of functionals of integral structure which are more complex than (0.2). For example, we consider functionals of the form

\[
\int_{-\infty}^{t} g(w(s)), \int_{-\infty}^{t} f(w(v)) dv, \int_{-\infty}^{t} h(\int_{-\infty}^{t} f(w(u)) du) dv) ds.
\]

The second class of functionals, considered in Sec. 3 of the present paper, is of the form

\[
B(t) = \int_{-\infty}^{\infty} f(y, \hat{t}(t, y)) dy, \quad (0.3)
\]

where \( f(y, x) \) is a continuous function. Functionals of this type appear in a natural manner at the consideration of the following problem. In the author's papers [5, 6] and in a paper by Kesten and Spitzer [7] one has obtained limit theorems for the sums

\[
S_N(t) = \sum_{k=1}^{[nt]} \eta_k,
\]

where \( \{\xi_k\}_{k=-\infty}^{\infty} \) are independent random variables, while \( \eta_k \) is a recurring random walk over the integral lattice, independent of the variables \( \xi_k \). If \( \eta_k \) is a walk with a finite variance and the distribution of the variables \( \xi_k \) belongs to the domain of attraction of the stable law with exponent \( \beta \) then, under an appropriate normalization, the processes \( S_n(t) \) converge weakly to the process \( S(t) \) which can be expressed in the form of the stochastic integral

\[
S(t) = \int_{-\infty}^{\infty} f(y) \hat{t}(t, y) dZ(y), \quad (0.4)
\]

where \( f(y) \) is a nonrandom function, while \( Z(y), Z^2(y), y > 0 \), are stable processes with exponent \( \beta \), independent from each other and from \( \hat{t}(t, y) \). For a process \( Z(y) \) with a...