
OCCUPATION TIMES FOR COUNTABLE MARKOV CHAINS. I.

CHAINS WITH DISCRETE TIME

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One obtains the analogues of the "Ray's description" of the local time for one-dimensional Brownian motion, valid for arbitrary homogeneous Markov chains with discrete time and countable state space. Contrary to the case of the Brownian motion, one establishes the absence of the Markov property for the process of the occupation time in the case of the simplest one-dimensional symmetric random walk.

1. Introduction

Let \( x(t), t = 0, 1, 2, ... \) be a homogeneous Markov chain with a finite or countable state space \( A \), let \( P_a \) be a probability measure for the process, starting at the point \( a \in A \), and let \( e \) be an integer-valued random variable with distribution

\[ P_e = e^{-t} \]

(\( B \) is a \( \sigma \)-algebra of events connected with the chain).

We denote by \( \tau(\xi) \) the occupation time of the process \( x(t) \) in the state \( \xi \in A \) to the moment \( e \):

\[ \tau(\xi) = \sum_{t=0}^{e} I_{\{x(t) = \xi\}}. \]

In Sec. 2 we give the description of the conditional finite-dimensional distribution of the random field \( \tau(\xi) \) under the condition \( x(e) = \xi \). More exactly, we give formulas which express the Laplace transforms of these finite-dimensional distributions in terms of the Green function of the Markov chain \( x(t) \). These formulas (and their derivation) are in a certain

sense similar to the "Ray's description" of the local time for a one-dimensional Brownian motion (see [1]). As it is known [1], the conditional local time for a Brownian motion (to the exponential moment $e$) is a Markov process. In the cases we consider, the state space $\mathbb{A}$ does not have any additional structure (dimension, order, etc.) and, therefore, one cannot talk of the Markov property and we are forced to consider all finite-dimensional distributions.

Nevertheless, in Sec. 4 we investigate the question of the Markov property for the special case of the simplest symmetric walk along the integral line. It turns out, somewhat unexpectedly, that even in this case, which is closest to a Brownian motion, the process $\tau(\xi)$ is not Markovian. As it will be proved in part II of our investigation, the absence of the Markov property is a consequence of the discrete time since, for the similar integral Markov process with continuous time, the process $\tau(\xi)$ is Markovian.

In Sec. 3 we describe a method for the computation of the Green function for integer-valued Markov sequences with a symmetric transition function, similar to the well-known representation of the Green function for ordinary differential equations (see [1]). In the case of "periodic with respect to the space" transition probabilities, one describes the explicit expression for the Green function.

Finally, in Sec. 5 there are given some additional formulas regarding the simplest symmetric walk.

In Part II we shall give results regarding chains with continuous time.

We mention that the occupation times for Markov chains have been investigated by a series of authors from other points of view; see, for example, [2-6].

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2. Finite-Dimensional Distribution of Occupation Times

We denote by $P(a, \xi)$ the transition function of our Markov chain $x(t)$ and by $P$ the corresponding transition operator in the space of functions on $\mathbb{A}$:

$$(Pf)(a) = \sum_{\xi} P(a, \xi) f(\xi).$$

We also set $q_0 = P - 1$.

We shall consider the discrete analogue of the heat-conduction equation

$$u(t+1, a) - u(t, a) = (q_0 u)(t, a)$$

with the initial condition $u(0, a) = f(a)$. It is understood that the operator $q_0$ acts on the variable $a$.

Obviously, for any initial condition $f$, Eq. (1) has a unique solution, given by the formula

$$u(t, a) = \sum_\xi P_t(a, \xi) f(\xi) = E_a f(x(t)).$$

Here $P_t$ is the transition function over $t$ steps and $E_a$ is the mathematical expectation under the condition $x(0) = a$. 