PERIODIC ORBITS IN A DYNAMICAL SYSTEM WITH THREE DEGREES OF FREEDOM

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Abstract. We use the analytical method of Lindstedt to make an inventory of the regular families of periodic orbits and to obtain approximate analytical solutions in a three-dimensional harmonic oscillator with perturbing cubic terms. We compare these solutions to the results of numerical computations at a specific orbital resonance.

1. Introduction

There have been numerous studies of dynamical systems with two degrees of freedom and the numerical (surfaces of sections ...) and analytical (adelphic integral, Lindstedt series, Birkhoff normalization ...) methods of investigation of these systems are now well established. On the other hand the dynamical properties of systems with three degrees of freedom are still poorly known and the methods of study of these systems deserve to be developed.

Martinet and Magnenat (1981) initiated the numerical exploration of the system of Hamiltonian

\[ H = \frac{1}{2}(x^2 + y^2 + z^2) + \frac{1}{2}(Ax^2 + By^2 + Cz^2) - \varepsilon xz^2 - \eta yz^2 \]  

(1)

by the method of invariant surfaces. This is a three dimensional harmonic oscillator with perturbing cubic terms. Further studies of this system (Magnevat, 1982; Martinet et al., 1981; Contopoulos et al., 1982; Hayli et al., 1983) are particularly interesting since the corresponding system with two degrees of freedom (where \( y = 0 \)) has already been studied in detail (Contopoulos, 1981, and references therein), especially near resonances, when \( \sqrt{A} \) and \( \sqrt{C} \) are commensurable.

In what follows we apply the Lindstedt method to this dynamical system to make an inventory of the regular families of periodic orbits, that is those that can be continued to the unperturbed system, and to obtain periodic solutions approximated by formal (not necessarily converging) trigonometric series. This method has already been applied successfully to the case of two degrees of freedom in the special case \( A = C \) (Presler and Broucke, 1981a, b) and in the general case of resonance and its neighborhood (Davoust, 1983). Three cases of resonances must be distinguished. We then compare our solutions to the more accurate numerical computations of Hayli et al. (1983) at the 6:4:3 resonance.

2. The Lindstedt Method

The equations of motion corresponding to (1) are

$$\begin{align*}
\ddot{x} + A\dot{x} &= \varepsilon z^2 e, \\
\ddot{y} + B\dot{y} &= \eta z^2 e, \\
\ddot{z} + Cz &= 2\varepsilon xze + 2\eta yze,
\end{align*}$$

(2)

where double dot stands for the second time \( t \) derivative and the right-hand sides (RHS) have been multiplied by the ordering parameter \( \varepsilon \) which indicates that the RHS are first-order perturbations. Indeed, the Lindstedt method is a perturbation method and is applicable only if the RHS are small relative to each term of the left-hand sides. In what follows \( \varepsilon \) will only be used to order the terms of the same magnitude. At the end of the calculations we set \( \varepsilon = 1 \).

At zeroth order the general solution of (2) is a combination of harmonic oscillations in \( x, y, \) and \( z \) of respective frequencies \( \sqrt{A}, \sqrt{B}, \) and \( \sqrt{C} \). The three frequencies must be commensurable for the motion to be periodic; this situation is called resonance. We set \( \sqrt{A} = wm, \sqrt{B} = wn, \) and \( \sqrt{C} = wl \), where \( m, n, \) and \( l \) are mutually prime numbers. The general solution is then periodic too, of frequency \( w \).

We now look for a periodic solution of (2) of frequency \( \Omega \) close to \( w \). To this end we develop the variables \( x, y, \) and \( z \), but also \( \Omega \), in power series of \( \varepsilon \).

$$\begin{align*}
\Omega^2 &= w^2(1 + \phi_1 e + \phi_2 e^2 + \phi_3 e^3 + \cdots) \\
x &= x_0 + x_1 e + x_2 e^2 + x_3 e^3 + \cdots
\end{align*}$$

and similar expansions for \( y \) and \( z \). The \( \phi_i \) are constants, the \( x_i, y_i, \) and \( z_i \) are periodic functions of time. We eliminate the frequency \( \Omega \) from the trigonometric terms by the change of time variable \( \tau = \Omega t \) and we denote by double prime the second derivative with respect to \( \tau \). Finally we denote by \( p \) and \( q \) the coefficients \( e \) and \( \eta \) divided by \( w^2 \). The equations of motion become:

to zeroth order in \( \varepsilon \):

$$\begin{align*}
x_0'' + m^2 x_0 &= 0, \\
y_0'' + n^2 y_0 &= 0, \\
z_0'' + l^2 z_0 &= 0,
\end{align*}$$

(3)

to first order:

$$\begin{align*}
x_1'' + m^2 x_1 &= -\phi_1 x_0'' + pz_0^2, \\
y_1'' + n^2 y_1 &= -\phi_1 y_0'' + qz_0^2, \\
z_1'' + l^2 z_1 &= -\phi_1 z_0'' + 2px_0 z_0 + 2qy_0 z_0,
\end{align*}$$

(4)