ON A THEOREM OF LOVÁSZ ON COVERS IN r-PARTITE HYPERGRAPHS

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A theorem of Lovász asserts that \( \tau(H)/\tau^*(H) \leq r/2 \) for every \( r \)-partite hypergraph \( H \) (where \( \tau \) and \( \tau^* \) denote the covering number and fractional covering number respectively). Here it is shown that the same upper bound is valid for a more general class of hypergraphs: those which admit a partition \( (V_1, \ldots, V_k) \) of the vertex set and a partition \( p_1 + \ldots + p_k \) of \( r \) such that \( |e \cap V_i| \leq p_i \leq r/2 \) for every edge \( e \) and every \( 1 \leq i \leq k \). Moreover, strict inequality holds when \( r > 2 \), and in this form the bound is tight. The investigation of the ratio \( \tau/\tau^* \) is extended to some other classes of hypergraphs, defined by conditions of similar flavour. Upper bounds on this ratio are obtained for \( k \)-colourable, strongly \( k \)-colourable and (what we call) \( k \)-partitionable hypergraphs.

1. Introduction

A hypergraph \( H \) is an ordered pair \( H = (V, E) \), where \( V = V(H) \) is a finite set (the set of vertices) and \( E = E(H) \) is a non-empty collection of non-empty subsets of \( V \) called edges. The set \( V \) is called the vertex set of \( H \), the set \( E \) is the edge set of \( H \). The rank of \( H \) is \( r(H) = \max \{|e| : e \in E(H)\} \). If all edges of \( H \) are of size \( r \), then \( H \) is \( r \)-uniform, or simply an \( r \)-graph. \( \binom{V}{r} \) denotes the hypergraph with vertex set \( V \) and edge set consisting of all subsets of \( V \) of size \( r \).

The set \( \{1, \ldots, n\} \) is denoted by \( [n] \).

A set \( T \subseteq V \) is called a cover (or a transversal) of the hypergraph \( H = (V, E) \) if \( T \cap e \neq \emptyset \) for every \( e \in E(H) \). The minimum cardinality of a cover of \( H \) is called the covering number of \( H \) and denoted by \( \tau(H) \). E.g., \( \tau([n]) = n - r + 1 \) for all positive integers \( n \geq r \).

A set \( M \subseteq E \) is called a matching in the hypergraph \( H = (V, E) \) if all edges of \( M \) are pairwise disjoint. The maximum cardinality of a matching in \( H \) is called the matching number of \( H \) and denoted by \( \nu(H) \).
Many problems of combinatorics can be formulated as the determination of the covering number of a hypergraph. The exact calculation of the covering number of an arbitrary hypergraph is known to be NP-hard. Hence the question of 'good' approximation of the covering number is of great importance. One of the simplest ways to estimate the covering number is by using the linear programming bound.

A fractional cover of the hypergraph \( H = (V, E) \) is a function \( g : V \to \mathbb{R}^+ \) such that \( \sum_{v \in e} g(v) \geq 1 \) for every \( e \in E(H) \). The value of the fractional cover \( g \) is \( |g| = \sum_{v \in V} g(v) \). The minimum of \( |g| \) over all fractional covers of \( H \) is the fractional covering number of \( H \), denoted by \( \tau^*(H) \).

Similarly, a fractional matching in \( H = (V, E) \) is a function \( f : E \to \mathbb{R}^+ \) such that \( \sum_{e \ni v} f(e) \leq 1 \) for every \( v \in V(H) \). The value of the fractional matching \( f \) is \( |f| = \sum_{e \in E} f(e) \). The maximum of \( |f| \) over all fractional matchings of \( H \) is the fractional matching number of \( H \), denoted by \( \nu^*(H) \).

For every hypergraph \( H \) one has: \( \tau(H) \geq \tau^*(H) \), \( \nu^*(H) \geq \nu(H) \). It is easy to see that the above two problems are in fact a pair of dual linear programming problems. The Duality Theorem of Linear Programming asserts that:

(i) for every fractional cover \( g \) and every fractional matching \( f \) one has: \( |g| \geq |f| \);
(ii) \( \tau^* = \nu^* \);
(iii) if \( g \) is an optimal fractional cover (i.e., \( |g| = \tau^* \)) and \( f \) is an optimal fractional matching (i.e., \( |f| = \nu^* \)), then:

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\begin{align*}
f(e) > 0 & \text{ implies } \sum_{v \in e} g(v) = 1, \\
g(v) > 0 & \text{ implies } \sum_{e \ni v} f(e) = 1.
\end{align*}
\]

(These are the so called complementary slackness conditions.)

**Example 1.** \( H = \binom{[n]}{r} \).

Define a fractional cover \( g : V \to \mathbb{R}^+ \) by \( g(v) = 1/r \) for every \( v \in V \) and a fractional matching \( f : E \to \mathbb{R}^+ \) by \( f(e) = 1 / \left( \binom{n-1}{r-1} \right) \) for every \( e \in E \). Then \( |g| = |f| = n/r \), so \( g \) and \( f \) are an optimal fractional cover and fractional matching, respectively, and \( \tau^* = \nu^* = n/r \).

As mentioned above, the fractional covering number may be used as an estimate for the covering number. It is natural to ask how good this estimate is, or, in other words, how large the ratio \( \tau/\tau^* \) can be for certain types of hypergraphs. A very useful upper bound on the ratio \( \tau/\tau^* \) was obtained independently by Lovász [3], Sapozhenko [6] and Stein [7]; this bound asserts that \( \tau/\tau^* \leq 1 + \log D \), where \( D = \max_{v \in V} |\{e : v \in e\}| \) — the maximum degree in the hypergraph \( H \).