Let $A_i = (a_{i1}, \ldots, a_{in})$, $1 \leq i \leq r$, denote $r$ sequences of real numbers and let $p_k$ ($1 \leq k \leq n$) be positive "weights" regarded as fixed. We shall write

$$A_n(A_1, \ldots, A_r) = \sum_{k=1}^{n} p_k a_{1k} \cdots a_{rk} - \left( \sum_{k=1}^{n} p_k a_{1k} \right) \ldots \left( \sum_{k=1}^{n} p_k a_{rk} \right) \left( \sum_{k=1}^{n} p_k \right)^{-1}.$$  

The classical results, associated with Čebyšev's name, may be summarized in the following two theorems:

**Theorem 1.** If $A_1, A_2$ are sequences of real numbers which are monotonic in the same sense, then

$$A_n(A_1, A_2) \geq 0$$

and the inequality is reversed if $A_1, A_2$ are monotonic in the opposite sense. In either case, $A_n(A_1, A_2) = 0$ if and only if $A_1$ or $A_2$ is constant.

A sequence is naturally said so be "constant" if all its terms are equal.

**Theorem 2.** If $r \geq 3$ and $A_1, \ldots, A_r$ are sequences of non-negative numbers which are monotonic in the same sense, then

$$A_n(A_1, \ldots, A_r) \geq 0.$$  

There is equality in (2) if and only if at least $r - 1$ sequences among $A_1, \ldots, A_r$ are constant.

A finite sequence $B = (b_1, \ldots, b_n)$ will be said to be increasing in mean (with respect to the given system of weights) if


Key words and phrases. Čebyšev's inequality, sequence monotonic in mean, refinement.
In [1], the following two theorems are given:

**THEOREM 3.** Let $A_1, A_2$ be two sequences of $n$ real numbers each. If they are monotonic in mean in the same sense, then (1) is valid, and if they are monotonic in mean in opposite senses, inequality (1) is reversed. In either case the equality occurs if and only if $A_1$ or $A_2$ is constant.

**THEOREM 4.** Let $r \geq 3$, and let $A_1, \ldots, A_r$ be sequences of non-negative numbers which are monotonic in mean in the same sense. Then (2) is valid. There is equality in (2) if and only if at least $r - 1$ sequences among $A_1, \ldots, A_r$ are constant.

**REMARK 1.** Using Theorems 1–4, we can easily prove the corresponding integral analogous.

Let

$$M(I) = P_I(A_I(AB, p) - A_I(A, p)A_I(B, p)),$$

where

$$P_I = \sum_{i \in I} p_i$$

and

$$A_I(A, p) = \frac{1}{P_I} \sum_{i \in I} p_i a_i.$$

[3] contains a general result wherefrom we obtain the following theorem:

**THEOREM 5.** Let $I$ and $J$ denote nonempty disjoint finite sets of distinct positive integers. Suppose that $(a_k); (b_k)$ and $(p_k)$ with $p_k > 0$ and $k \in I \cup J$, are sequences of real numbers. If the pairs

$$(4) \quad (A_I(A, p), A_J(A, p)) \text{ and } (A_I(B, p), A_J(B, p))$$

are similarly ordered, then

$$M(J \cup I) \geq M(I) + M(J).$$

If the pairs (4) are oppositely ordered, then the sense of (5) reverses. Equality holds if and only if either $A_I(A, p) = A_J(A, p)$ or $A_I(B, p) = A_J(B, p)$.