ON AN APPLICATION OF
HALL’S REPRESENTATIVES THEOREM
TO A FINITE GEOMETRY PROBLEM

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Abstract

It is well-known that n points not belonging to a hyperplane determine at least n hyperplanes. The possible configurations of hyperplanes in the case when the number of hyperplanes is equal to n are known, too. In this paper we obtain these results by means of Hall’s representatives theorem. The setting is that of a finite geometry.

1. Introduction

Let $X$ be a finite set and let $P(X)$ denote the family of all subsets of $X$. Every two-element subset of $X$ will be called a pair. We will say that a family $\mathcal{L} \subseteq P(X)$ determines a lines structure on $X$, if the following conditions hold:

$(\mathcal{L} - 1)$ card $l \geq 2$ for $l \in \mathcal{L},$

$(\mathcal{L} - 2)$ every pair of elements of $X$ occurs in a unique $l \in \mathcal{L}$.

The pair $(X, \mathcal{L})$ will be called a geometry of lines, the elements of $X$ are called points; members of $\mathcal{L}$ are called lines. In $[3]$ $(X, \mathcal{L})$ is called a finite linear space. If $x \in l, l \in \mathcal{L}$, then we will say that $x$ lies on $l$ or that $l$ passes through $x$. The line determined by two distinct points $x, y \in X$ will be denoted by $l(x, y)$.

De Bruijn and Erdős showed in $[2]$ that if card $\mathcal{L} > 1$ then card $\mathcal{L} \geq \text{card } X$. Moreover, if card $\mathcal{L} = \text{card } X = n > 1$, then only two realizations of lines geometry are possible, namely either a near-pencil (one line contains all but one of the points) or a projective plane and then each line in $\mathcal{L}$ has $k$ elements, exactly $k$ lines pass through each point and $n = k(k - 1) + 1$.

Let $(X, \mathcal{L})$ be a lines geometry. A subset $F$ of $X$ will be called a linear variety if the following condition is satisfied:

$(\mathcal{L} - 3)$ $x, y \in F, x \neq y \Rightarrow l(x, y) \subseteq F.$

We say that a subset $M$ of $X$ is collinear if $M \subseteq l$ for some $l \in \mathcal{L}$. Otherwise $M$ will be called noncollinear. Each three-element subset of $X$ will be called a triple.


Key words and phrases. Finite incidence structure, Hall’s theorem, configuration of planes.
We shall say that a family \( S \subseteq \mathcal{P}(X) \) determines a *structure of planes* on \( X \) if the following conditions hold:

- \((S - 1)\) each plane \( \pi \in S \) includes a noncollinear triple,
- \((S - 2)\) each noncollinear triple belongs to exactly one plane,
- \((S - 3)\) each plane \( \pi \in S \) is a linear variety.

The triple \((X, \mathcal{L}, S)\) will be called a *planes geometry*. In the case when \( \text{card } S > 1 \), \((X, \mathcal{L}, S)\) will be called a *nontrivial planes geometry*.

In this paper we give two theorems concerning the nontrivial planes geometry \((X, \mathcal{L}, S)\), which are related to the above quoted results of de Bruijn and Erdős. Namely, we prove that \( \text{card } S \geq \text{card } X \), and moreover, in the case when \( \text{card } S = \text{card } X \), we indicate possible configurations of planes.

Similar results have already been obtained in [1] and [5] in terms of lattice and matroid theory, respectively. However, our approach seems to be elementary and the application of Hall’s representatives theorem [4] appears to be interesting.

The reader is referred to [6], where in the references to Problem 27 a wide information on related questions can be found.

### 2. The results

In this section we first give a few simple properties of \((X, \mathcal{L}, S)\), and next prove the main theorems.

**PROPERTY 1.** The set of all lines contained in \( \pi \) determines a lines structure on \( \pi \).

**PROPERTY 2.** The intersection of any number of planes is either the empty set, or a point, or a line.

**PROPERTY 3.** Each \( l \in \mathcal{L} \) and \( x \notin l \) determine exactly one plane, which we will denote by \( \pi(l, x) \).

**PROPERTY 4.** Any two intersecting lines determine exactly one plane.

**PROPERTY 5.** If \( l_1, l_2 \) do not belong to one plane and \( x, y \in l_1 \), \( x \neq y \), then \( y \notin \pi(l_2, x) \).

**PROPERTY 6.** If for a lines geometry \((X, \mathcal{L})\) we have \( \text{card } \mathcal{L} > 1 \), then for each \( l \in \mathcal{L} \) there exists \( x \notin l \) and for each \( y \) there exists \( l \) such that \( y \notin l \).