STRONGLY REGULAR RINGS AND RATIONAL IDENTITIES OF DIVISION RINGS

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Abstract

We look at division rings in the variety of strongly regular rings and show a connection to the study of rational identities on division rings.

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In this paper all rings have a multiplicative identity; thus, the first-order language of rings consists of the usual logical apparatus along with two binary operations (addition and multiplication), a unary operation (additive inverse) and two nullary operations (the additive and multiplicative identities). A ring in which the sentence

$$\forall x \exists y [x^2 y = x \land y^2 x = y]$$

holds is said to be strongly regular. It has been pointed out by several authors (see for example [5]) that in a strongly regular ring, the $y$ in the sentence is uniquely determined by the $x$; hence, we may look on the map $x \rightarrow y$ as a unary operation and in this way the class of strongly regular rings is a variety in the classical sense. For convenience, let us denote the $y$ as a function of $x$ by $x^q$, ($q$ for quasi-inverse). A subclass is a subvariety if it is defined by a set of identities in the language of strongly regular rings, that is, by a set of strongly regular identities. If $R$ is a strongly regular ring let $V(R)$ denote the subvariety generated by $R$.

If $D$ is a division ring, then

$$x^q = \begin{cases} x^{-1} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We refer the reader to [2] for a discussion of rational identities on division rings. We may look on the rational polynomials as formal polynomials in

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variables $x_1, x_2, \ldots$ in the language $\{0, 1, +, -, \cdot, -1\}$, that is, the language of rings together with a new unary operation $()^{-1}$. When we specialize these polynomials down to particular division rings, the operator "$()^{-1}$" is only a partial operation in the sense that $0^{-1}$ is not defined. A rational polynomial is said to be a rational identity on a division ring if it vanishes whenever under a specialization, all of its “parts” are defined.

The study of strongly regular identities on division rings and the study of rational identities on division rings are related but are certainly not identical. Every strongly regular identity gives rise to a rational identity, but the converse does not hold. However, work done on rational identities on division rings allow us to say a great deal about the strongly regular identities.

**Theorem 1.** Associated with each rational polynomial is a strongly regular polynomial with the property that the former is a rational identity on a division ring if and only if the latter is a strongly regular identity on the division ring.

Thus $V(D) \subset V(C)$ implies that $D$ satisfies all the rational identities satisfied by $C$.

**Proof.** Let $P$ be a rational polynomial. Suppose that $P_1, P_2, \ldots, P_k$ are the polynomial parts of $P$ for which $P_i^{-1}$ occurs in $P$ ($i = 1, 2, \ldots, k$). We now construct a strongly regular polynomial $P^*$ from $P$ as follows. In $P_i$ replace the symbol ‘$-1$’ by the symbol ‘$q$’. This gives us the strongly regular polynomial $P'$. Let $P^*$ be the polynomial

$$P'P_1'(P_1')^qP_2'(P_2')^q \ldots P_k'(P_k')^q.$$

If $D$ is a division ring, then $P$ and $P^*$ are equal under any specialization in which $P$ is defined, and $P^*$ is zero if $P$ is undefined under a specialization. Therefore $P$ is a rational identity on $D$ if and only if $P^*$ is a strongly regular identity on $D$.

The above map $*$ allows us to consider rational identities as a subset of strongly regular identities.

**Theorem 2.** Suppose that the fields $F$ and $F'$ are finite dimensional extensions of the rational numbers. Then $F$ and $F'$ satisfy the same strongly regular identities if and only if they are isomorphic.

**Proof.** If $F$ and $F'$ are isomorphic, then they satisfy the same strongly regular identities.

Suppose that $F$ and $F'$ satisfy the same strongly regular identities and that $F = Q(\alpha)$. Let $p(x)$ be a minimal polynomial for $\alpha$ with coefficients in the integers. Then $F$ does not satisfy the identity $p(x)[p(x)]^q - 1 = 0$. Thus