AN APPLICATION OF GOMORY CUTS IN NUMBER THEORY

B. VIZVÁRI (Budapest)

Abstract

Frobenius has stated the following problem. Suppose that $a_1, a_2, \ldots, a_n$ are given positive integers and $\text{g.c.d.} (a_1, \ldots, a_n) = 1$. The problem is to determine the greatest positive integer $g$ so that the equation

$$\sum_{i=1}^{n} a_i x_i = g$$

has no nonnegative integer solution. Showing the interrelation of the original problem and discrete optimization we give lower bounds for this number using Gomory cuts which are tools for solving discrete programming problems.

In the first section an important theorem is cited after some remarks. In Section 2 we state a parametric knapsack problem. The Frobenius problem is equivalent with finding the value of the parameter where the optimal objective function value is maximal. The basis of this reformulation is the above mentioned theorem. Gomory's cutting plane method is applied for the knapsack problem in Section 3. Only one cut is generated and we make one dual simplex step after cutting the linear programming optimum of the knapsack problem. Applying this result we gain lower bounds for the Frobenius problem in Section 4. In the last section we show that the bounds are sharp in the sense that there are examples with arbitrary many coefficients where the lower bounds and the exact solution of the Frobenius problem coincide.

1. Introduction

It is very easy to prove the following theorem.

THEOREM 1.1. Let $a_1, \ldots, a_n$ be positive integers and $\text{g.c.d.} (a_1, \ldots, a_n) = 1$. Consider the linear form

$$\sum_{i=1}^{n} a_i x_i$$

where the variables are nonnegative integers. There exists an integer $g$ such that the linear form takes every integer value which is greater than $g$.

AMS (MOS) subject classification (1980). Primary 10B05.
Key words and phrases. Coin changing problem, Frobenius problem, Gomory cut, lower bound.
The Frobenius problem is to determine the smallest number for which the statement of Theorem 1.1 is true. We shall denote this number by \( g(a_1, \ldots, a_n) \).

A lot of papers deal with the Frobenius problem. The research has two main directions: (a) exact solution of special problems, (b) general upper bounds. As far as the author knows, only two lower bounds have been proved so far. The first one has appeared in [4]. This lower bound is

\[
V = \sqrt[n-1]{a_1 \ldots a_n} - \sum_{i=1}^{n} a_i.
\]

The second one is a general technic to generate lower and upper bounds [1]. A rich list of references is in [6] and the reader can find the best review in [5].

As the Frobenius problem is NP-hard in any application of it, the simple, quickly computable lower and upper bounds can have great importance.

We shall give some new lower bounds. To do this we need an exact formula of the general case. This result was published in [2]. We use the following notations.

\[
\mathcal{F} = \left\{ y : \exists x \in \mathbb{Z}_+^n ; y = \sum_{i=1}^{n} a_i x_i \right\}
\]

where \( \mathbb{Z}_+^n \) is the set of the \( n \)-dimensional nonnegative integer vectors. We say for a positive integer \( y \) that it has a representation if \( y \in \mathcal{F} \) and the appropriate vector \( x \) is the representation of \( y \).

**Theorem 1.2.** Let the numbers \( a_1, \ldots, a_n \) be as above and \( k \) an arbitrary index \( 1 \leq k \leq n \). Denote \( t_l \) the smallest number of residue class \( l \) (mod \( a_k \)) which has a representation, where \( 1 \leq l \leq a_k - 1 \). Now

\[
g(a_1, \ldots, a) = \max_l t_l - a_k.
\]

2. Reformulation of the Frobenius problem
   as a parametric knapsack problem

We shall use the previous notations.

**Remark 2.1.** In every representation of \( t_l, x_k = 0 \), where \( l \) is an arbitrary residue class. Namely in the opposite case there exists such a representation where \( x_k \geq 1 \). Therefore \( t_l - a_k \) which belongs to the same residue class, also has a representation. This contradicts the minimality of \( t_l \).

Till now we have only supposed that the greatest common divisor of the positive integers \( a_1, \ldots, a_n \) is equal to 1. Let the smallest one of the num-