A REMARK ON THE ALTERNATING ALGORITHM

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Introduction

Let us recall the definition of Čebyšev subspaces of a normed linear space \((X, || . ||)\) \([2]\).

Let \((Y, || . ||)\) be a closed subspace of the real normed linear space \((X, || . ||)\). We say that \(Y\) is a Čebyšev subspace if for all \(x \in X\), the following set contains exactly one element:

\[
\{y_0 \in Y; ||x - y_0|| = \inf_{y \in Y} ||x - y||\}.
\]

In this case, we denote \(y_0\) by \(P_Y(x)\).

Concerning the mapping \(P\), the following algorithm is well known. Denoting the closed linear hull of the closed subspaces \(U\) and \(V\) by \(\overline{\{U, V\}}\),

we can try to compute \(P_{\overline{\{U, V\}}}(x)\) in the following way: We set \(K = (I - P_U)(I - P_V)\) and hope that

\[
\lim_{n \to \infty} K^n(x)
\]

exists, and

\[
x - P_{\overline{\{U, V\}}}(x) = \lim_{n \to \infty} K^n(x).
\]

The above way of computing is called the alternating algorithm. There are many special cases when the convergence of this algorithm is proved.

For example, convergence is proved in the case when \((X, || . ||)\) is a Hilbert space, or, more generally, in a special class of uniformly convex and smooth Banach spaces \([1]\).

In this paper, we prove the convergence in an arbitrary uniformly convex uniformly smooth Banach space. (We remark that in such spaces all closed subspaces are Čebyšev.)

Key words and phrases. Čebyšev subspace, metric projection.
The result

**Theorem.** Let \((X, \| \cdot \|)\) be a uniformly smooth, uniformly convex real normed linear space with closed subspaces \(U\) and \(V\). Introducing the notation
\[
K = (I - P_U)(I - P_V),
\]
for all \(x \in X\), we have
\[
\lim_{n \to \infty} K^n(x) = x - P_{\overline{U \cup V}}(x).
\]

**Proof.** We need the following, elementary

**Proposition.** For any subspace \(W \subset X\) and any element \(y \in X\),
\[
\|(I - P_W)y\| \leq \|y\|.
\]

**Proof of the Proposition.**
\[
\|(I - P_W)y\| = \min_{w \in W} \|y - w\| \leq \|y - 0\| = \|y\|.
\]

Returning to the proof of the theorem, let us assume first that \(x \notin \{U, V\}\). Using the Proposition, \(\|K^n(x)\|\) is monotonically decreasing, so, convergent.

On the other hand,
\[
x - (I - P_V)(x) \in \overline{U \cup V},
\]
\[
x - (I - P_V)(x) + (I - P_V)(x) - (I - P_U)(I - P_V)(x) \in \overline{U \cup V},
\]
and this implies \(x - Kx \in \overline{U \cup V}\).

Using this fact,
\[
x - K^n(x) = x - K(x) + K(x) - K^2(x) + \ldots - \ldots +
\]
\[
+ K^{n-1}(x) - K^n(x) \in \overline{U \cup V}.
\]

Let us consider an \(N(\varepsilon)\) such that for all \(n > N(\varepsilon)\),
\[
\|K^n(x)\| - \lim_{k \to \infty} \|K^k(x)\| < \varepsilon
\]
where \(\varepsilon > 0\) fixed.

Now, for \(n > N(\varepsilon)\), the proposition implies
\[
\|K^n(x)\| < 2\varepsilon.
\]

We show now that
\[
\|P_V(K^n(x))\| < \varepsilon_1
\]
for arbitrary \(\varepsilon_1 > 0\), when we choose \(\varepsilon\) sufficiently small.