THE TREE NUMBER OF A GRAPH WITH A GIVEN GIRTH

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Abstract

A family of subtrees of a graph $G$ whose edge sets form a partition of the edge set of $G$ is called a tree decomposition of $G$. The minimum number of trees in a tree decomposition of $G$ is called the tree number of $G$ and is denoted by $\tau(G)$. It is known that if $G$ is connected then $\tau(G) \leq \lceil |G|/2 \rceil$. In this paper we show that if $G$ is connected and has girth $g \geq 5$ then $\tau(G) \leq \lceil |G|/g \rceil + 1$. Surprisingly, the case when $g = 4$ seems to be more difficult. We conjecture that in this case $\tau(G) \leq |G|/4 + 1$ and show a wide class of graphs that satisfy it. Also, some special graphs like complete bipartite graphs and $n$-dimensional cubes, for which we determine their tree numbers, satisfy it. In the general case we prove the weaker inequality $\tau(G) \leq \lceil (|G| - 1)/3 \rceil + 1$.

1. Introduction

In the paper we consider undirected graphs without loops and multiple edges. Our terminology follows Harary [6].

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A tree decomposition is a special case of an edge decomposition of a graph. Many results have been obtained in the area of decompositions, extensive lists of references can be found in the survey papers [1] and [3]. Among other things various decomposition parameters defined analogously to the tree number like e.g., the chromatic index (the minimum number of matchings required to decompose a graph), the arboricity (the minimum number of forests required to decompose a graph), or the linear arboricity (the minimum number of forests whose each component is a path required to decompose a graph) have been studied, and estimates and, in some cases, exact formulas for them have been found (see, e.g., [4], [8] and [5]).

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For the tree number of a connected graph $G$ we have the following bounds (if $G$ is not connected then its tree number is the sum of tree numbers of its connected components, hence, it suffices to restrict to connected graphs only):

\[
\left| |E(G)|/(|G| - 1) \right| \leq \tau(G) \leq \lfloor |G|/2 \rfloor.
\]

Here and in the sequel by $|G|$ we denote the number of vertices in $G$. The lower bound is obvious (no tree in a tree decomposition of $G$ can have more than $|G| - 1$ edges) and the upper one was found by Chung [2].

In this paper we study the tree number of a connected graph $G$ with girth $g$. In the case when $g \geq 5$, it is not hard to show that $\tau(G) \leq \lfloor |G|/g \rfloor + 1$ (Theorem 3). However, the case $g = 4$ (i.e. of triangle-free graphs) seems to be much more difficult. Although there is a wide class of connected triangle-free graphs $G$ for which we prove $\tau(G) \leq \lfloor |G|/4 \rfloor + 1$ (Theorem 4) and although some special triangle free graphs like $n$-dimensional cubes and complete bipartite graphs, for which we determine their tree numbers (Theorems 5 and 6), also satisfy it, for an arbitrary connected triangle-free graph $G$ we can only prove a weaker bound $\tau(G) \leq \lfloor (|G| - 1)/3 \rfloor + 1$ (Theorem 7).

Our results and proofs are contained in the next section. We conclude the introduction with two auxiliary lemmas and some notation.

**Lemma 1.** Let $G$ be a connected graph, let $e \in E(G)$ be a bridge of $G$ and let $G_1$ and $G_2$ be the two components of $G - e$. Then $\tau(G) = \tau(G_1) + \tau(G_2) - 1$.

The proof is evident and we omit it.

**Lemma 2** (see e.g., [7], Problem 6.8a, p. 40). Let $G$ be a 2-connected graph of order $n$, let $u \in V(G)$ and let $n = n_1 + n_2$ for some positive integers $n_1$ and $n_2$. Then $V(G)$ has a partition \{ $S_1, S_2$ \}, such that $|S_i| = n_i$, $i = 1, 2$, $S_1$ and $S_2$ induce connected subgraphs and $u \in S_2$.

Through the paper, $K_n$ denotes a complete graph with $n$ vertices, $K_{m,n}$ denotes a complete bipartite graph with vertex classes having $m$ and $n$ vertices, $C_n$ and $P_n$ denote a cycle and a path with $n$ edges, respectively. By $g(G)$ we denote the girth of $G$ and, for a subset $W$ of the vertex set of $G$, by $G[W]$ we denote the subgraph of $G$ induced by $W$.

2. Results

Our first result gives an upper bound for the tree number of a connected graph with girth at least 5.