EXPLICIT EXPRESSION FOR THE TRANSITION DENSITY OF THE SOLUTION OF A STOCHASTIC DIFFUSION EQUATION WITH PIECEWISE-CONSTANT DRIFT COEFFICIENT

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The explicit form of the transition density is determined for the solution $\xi(t)$ of the stochastic diffusion equation $d\xi(t) = a(\xi(t))dt + dw(t)$, where $a(z) = \alpha$ for $z \in [a, b]$ and $a(z) = 0$ for $z \notin [a, b]$, $w(t)$ is a Wiener process.

Consider the stochastic differential equation

$$d\xi(t) = a(\xi(t))dt + dw(t), \quad t > 0,$$

where $a(x) = \alpha I_{[a,b]}(x)$, $0 < a < b$, and $w(t)$ is a Wiener process defined on the probability space $(\Omega, F, P)$. Let $\xi(0) = x$.

In this paper, we derive an explicit expression for the transition probability density $p(0, x, t, y)$ of the diffusion process $\xi(t)$. A similar problem with piecewise-constant diffusion coefficient and zero drift coefficient was considered in [2].

By [3], the function $U(t, x) = M_t g(\xi(s))$ defined for $0 \leq t \leq s$ and $x \in (-\infty, \infty)$ satisfies the Kolmogorov equation

$$\frac{\partial U(t, x)}{\partial t} + \alpha x \frac{\partial U(t, x)}{\partial x} + \frac{1}{2} \frac{\partial^2 U(t, x)}{\partial x^2} = 0$$

with the boundary condition

$$\lim_{t \to s} U(t, x) = g(x),$$

where $g(x)$ is a twice continuously differentiable function. Denote $V(t, x) = U(s - t, x)$. Then Eq. (2) for our case takes the form

$$\frac{\partial V(t, x)}{\partial t} + \alpha x \frac{\partial V(t, x)}{\partial x} + \frac{1}{2} \frac{\partial^2 V(t, x)}{\partial x^2} = 0,$$

with the initial condition $V(0, x) = g(x)$.

We use the Laplace transform to solve problem (3). Multiplying both sides of system (3) by $e^{-\lambda t}$ and integrating over $t$ from 0 to $+\infty$, we obtain respectively two ordinary linear differential equations of second order with constant coefficients:

$$\frac{d^2 \tilde{V}(\lambda, x)}{dx^2} - 2\lambda \tilde{V}(\lambda, x) = -2g(x), \quad x \leq a, \quad x > b,$$

$$\frac{d^2 \tilde{V}(\lambda, x)}{dx^2} + 2\alpha \frac{d \tilde{V}(\lambda, x)}{dx} - 2\lambda \tilde{V}(\lambda, x) = -2g(x), \quad a \leq x \leq b,$$

where $\overline{V}(\lambda, x)$ is the Laplace transform of $V(t, x)$. Solving Eqs. (4) and using the conditions

1) $|\overline{V}(\lambda, x)| \leq c$

and

2) \[
\begin{aligned}
\overline{V}(\lambda, a - 0) &= \overline{V}(\lambda, a + 0) \\
\overline{V}(\lambda, b - 0) &= \overline{V}(\lambda, b + 0) \\
\overline{V}_x(\lambda, a - 0) &= \overline{V}_x(\lambda, a + 0) \\
\overline{V}_x(\lambda, b - 0) &= \overline{V}_x(\lambda, b + 0)
\end{aligned}
\]

we find that:

for $x < a$,

\[
\overline{V}(\lambda, x) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\lambda}} \times
\]

\[
\times \left[ e^{-\lambda x} \sqrt{2\lambda} \frac{ae^{-(\alpha - \beta) x} \sqrt{2\lambda} (e^{-(\beta - \alpha)} \sqrt{2\lambda + \alpha^2} - e^{-(\beta - \alpha)} \sqrt{2\lambda + \alpha^2})}{Re(\beta - \alpha) \sqrt{2\lambda + \alpha^2} - R_1 e^{-\alpha} \sqrt{2\lambda - \alpha^2}} ight] g(y) dy +
\]

\[
+ \int_{a}^{b} \frac{e^{-\lambda x} \sqrt{2\lambda}}{\sqrt{2\lambda}} \frac{e^{\alpha y - \alpha} (R - \alpha) (e^{\beta y} \sqrt{2\lambda + \alpha^2} - e^{\beta y} \sqrt{2\lambda + \alpha^2})}{Re(\alpha - \beta) \sqrt{2\lambda + \alpha^2} - R_1 e^{-\alpha} \sqrt{2\lambda - \alpha^2}} g(y) dy +
\]

\[
+ 2 \int_{b}^{\infty} \frac{e^{-\lambda x} \sqrt{2\lambda}}{\sqrt{2\lambda}} \frac{e^{\alpha y - \alpha} \sqrt{2\lambda + \alpha^2} - e^{\alpha y - \alpha} \sqrt{2\lambda + \alpha^2}}{Re(\alpha - \beta) \sqrt{2\lambda + \alpha^2} - R_1 e^{-\alpha} \sqrt{2\lambda - \alpha^2}} g(y) dy;
\]

for $a \leq x \leq b$,

\[
\overline{V}(\lambda, x) = \int_{-\infty}^{a} \frac{e^{\alpha(x - y)}}{\sqrt{2\lambda}} \times
\]

\[
\times \left[ e^{-\lambda x} \sqrt{2\lambda} \frac{(R - \alpha) e^{\beta y} \sqrt{2\lambda + \alpha^2} - (R_1 - \alpha) e^{\beta y} \sqrt{2\lambda + \alpha^2}}{Re(\alpha - \beta) \sqrt{2\lambda + \alpha^2} - R_1 e^{-\alpha} \sqrt{2\lambda - \alpha^2}} \right] g(y) dy +
\]

\[
+ \int_{a}^{b} \frac{e^{-\lambda x} \sqrt{2\lambda}}{\sqrt{2\lambda}} \frac{1}{\sqrt{2\lambda}} \times
\]

\[
\times \left[ e^{\alpha(y - x)} \sqrt{2\lambda + \alpha^2} - e^{\alpha y - \alpha} \sqrt{2\lambda + \alpha^2} \right] \left[ e^{\alpha(y - x)} \sqrt{2\lambda + \alpha^2} - e^{\alpha y - \alpha} \sqrt{2\lambda + \alpha^2} \right] \left[ e^{\alpha(y - x)} \sqrt{2\lambda + \alpha^2} - e^{\alpha y - \alpha} \sqrt{2\lambda + \alpha^2} \right] g(y) dy +
\]

\[
+ \int_{b}^{\infty} \frac{e^{\alpha(y - x)} \sqrt{2\lambda}}{\sqrt{2\lambda}} \frac{e^{\alpha(y - x)} \sqrt{2\lambda + \alpha^2} - e^{\alpha y - \alpha} \sqrt{2\lambda + \alpha^2}}{Re(\alpha - \beta) \sqrt{2\lambda + \alpha^2} - R_1 e^{-\alpha} \sqrt{2\lambda - \alpha^2}} g(y) dy;
\]

for $x > b$,

\[
\overline{V}(\lambda, x) = 2 \int_{-\infty}^{a} \frac{e^{-(\alpha - \beta) x + \beta x} \sqrt{2\lambda}}{\sqrt{2\lambda}} \frac{e^{\alpha(y - x)} \sqrt{2\lambda + \alpha^2} - e^{\alpha y - \alpha} \sqrt{2\lambda + \alpha^2}}{Re(\alpha - \beta) \sqrt{2\lambda + \alpha^2} - R_1 e^{-\alpha} \sqrt{2\lambda - \alpha^2}} g(y) dy +
\]

\[
+ \int_{a}^{b} \frac{e^{-(\alpha - \beta) x} \sqrt{2\lambda}}{\sqrt{2\lambda}} \frac{e^{\alpha(y - x)} \sqrt{2\lambda + \alpha^2} - e^{\alpha y - \alpha} \sqrt{2\lambda + \alpha^2}}{Re(\alpha - \beta) \sqrt{2\lambda + \alpha^2} - R_1 e^{-\alpha} \sqrt{2\lambda - \alpha^2}} g(y) dy +
\]

\[
+ \int_{b}^{\infty} \frac{1}{\sqrt{2\lambda}} \left[ e^{-(\alpha - \beta) x} \sqrt{2\lambda} + e^{\alpha(y - x)} \sqrt{2\lambda + \alpha^2} - e^{-(\beta - \alpha) \sqrt{2\lambda + \alpha^2}} \right] \left[ e^{-(\beta - \alpha) \sqrt{2\lambda + \alpha^2}} - e^{-(\beta - \alpha) \sqrt{2\lambda + \alpha^2}} \right] g(y) dy.
\]