A feedback control system is considered. The transient time is estimated using the Lyapunov function. An algorithm for optimizing the estimate in the given class of functions is proposed.

Consider the feedback control system
\[
\dot{x} = Ax + b f(\sigma), \quad \dot{\sigma} = c^T x - \rho f(\sigma), \tag{1}
\]
where b, c, x are n-dimensional vectors, A is an asymptotically stable matrix, \( \rho > 0 \) is a scalar, \( f(\sigma) \) is a nonlinear characteristic satisfying the condition
\[
m\sigma^2 \leq f(\sigma) \leq M\sigma^2, \quad f(0) = 0; \tag{2}
\]
m and M are given positive constants. The function \( f(\sigma) \) characterizes the feedback and is usually not known exactly. The parameters of the control system are chosen so as to ensure asymptotic stability of the zero solution of system (1) for an arbitrary function \( f(\sigma) \) of the form (2) \cite{1, 3-5}.

Note that stability alone is insufficient for the synthesis of control systems. We should also consider the operating performance characteristics of the systems. One of the main performance measures is the duration of the control process, i.e., the time it takes the transient to become a steady process with prespecified accuracy.

One of the methods for estimating the transient time is provided by the method of Lyapunov functions \cite{2}. Suppose that for the system of equations
\[
\dot{y} = F(y), \quad F(0) = 0 \tag{1^*}
\]
a function \( v(y) \) exists such that
\[
\lambda_1 \|y\|^2 \leq v(y) \leq \lambda_2 \|y\|^2, \quad \lambda_1 > 0, \quad \lambda_2 > 0,
\]
and its total derivative by system (1*) satisfies
\[
\dot{v}(y) \leq -\lambda_3 \|y\|, \quad \lambda_3 > 0.
\]
Then the time it takes the solution \( y = y(t) \) of system (1*) satisfying the condition \( y(0) = y_0 \) to reach the \( \varepsilon \)-neighborhood of the origin is bounded by
\[
T \leq \frac{\lambda_2}{\lambda_3} \ln \left[ \frac{\lambda_2}{\lambda_1} \frac{\|y_0\|^2}{\varepsilon^2} \right]. \tag{3}
\]
We see from (3) that the transient time bound essentially depends on \( \lambda_1, \lambda_2, \lambda_3 \), i.e., on the function \( v(y) \), which is nonunique. It is therefore desirable to find the Lyapunov function which produces the most accurate estimate of the transient time.

For system (1), the Lyapunov function is constructed as a quadratic form plus an integral of the nonlinearity \cite{4, 5}:
\[
v(x, \sigma) = (x^T, Hx) + \int_{\sigma}^{\sigma} f(\sigma) d\sigma. \tag{4}
\]
The total derivative by (1) has the form

\[
\dot{v}(x, \sigma) = -(x^*, f(\sigma)) \begin{bmatrix}
-(A^TH + HA) - (Hb + \frac{1}{2}c) \\
-(Hb + \frac{1}{2}c)^T 
\end{bmatrix} \begin{bmatrix}
x \\
f(\sigma)
\end{bmatrix},
\]

and the conditions of absolute stability stipulate the existence of a positive definite matrix \(H\) for which \(A^TH + HA\) is negative definite and satisfies the inequality \([2, 3]\)

\[-(Hb + \frac{1}{2}c)^T(A^TH + HA)^{-1}(Hb + \frac{1}{2}c) \leq \rho. \tag{5}\]

Assume that the function \(f(\sigma)\) satisfies conditions (2). Then for \(v(x, \sigma)\) of the form (4) we have the bound

\[\lambda_1(\|x\|^2 + \sigma^2) \leq v(x, \sigma) \leq \lambda_2(\|x\|^2 + \sigma^2),\]

where \(\lambda_1 = \min\{\lambda_{\text{min}}(H), \frac{1}{2}m\}, \lambda_2 = \max\{\lambda_{\text{max}}(H), \frac{1}{2}M\}, \lambda_{\text{max}}(H)\) and \(\lambda_{\text{min}}(H)\) are the maximum and the minimum eigenvalues of the matrix \(H\). For the total derivative of \(v(x, \sigma)\) by (1) we have

\[
\dot{v}(x, \sigma) \leq -\lambda_2(\|x\|^2 + \sigma^2),
\]

\[
\lambda_2 = \min\{1, M^2\} \lambda_{\text{min}}(C_1), \quad C_1 = \begin{bmatrix}
-(A^TH + HA) - (Hb + \frac{1}{2}c) \\
-(Hb + \frac{1}{2}c)^T 
\end{bmatrix}.
\]

**Definition.** The Lyapunov function

\[v_0(x, \sigma) = (x^T, H_0x) + \int_0^\sigma f(\sigma) d\sigma\]

is called optimal for transient time estimation if

\[H_0 = \arg \min_{H \in G(H)} \{\varphi(H)\}, \quad \varphi(H) = \frac{\lambda_2(H)}{\lambda_1(H)} \ln \left[\frac{\lambda_2(H)}{\lambda_1(H)}\right],\]

where \(G(H)\) is the set of positive definite matrices for which \(A^TH + HA\) is negative definite and satisfies

\[-(Hb + \frac{1}{2}c)^T(A^TH + HA)^{-1}(Hb + \frac{1}{2}c) \leq \rho.\]

Finding the Lyapunov function \(v_0(x, \sigma)\) is a complicated mathematical programming problem and its solution in general is quite difficult. To simplify the problem, we assume that the conditions of absolute stability (5) are satisfied for some matrix \(H_1\), i.e., \(H_1 \in G(H)\), and consider the class of matrices \(\alpha H_1\), \(\alpha > 0\). The membership \(\alpha H_1 \in G(H)\) is clearly determined by the inequality

\[-\frac{1}{\alpha} \left(\alpha H_1 b + \frac{1}{2}c\right)^T(A^T H_1 + H_1 A)^{-1}\left(\alpha H_1 b + \frac{1}{2}c\right) \leq \rho,\]

or

\[-\alpha^2 a^2 + \alpha b - \frac{1}{4}c^2 > 0,\]

where \(a^2 = -(H_1 b) (A^T H_1 + H_1 A)^{-1}(H_1 b) > 0, \quad b = c^T(A^T H_1 + H_1 A)^{-1} H_1 b + \rho > 0, \quad c = -c^T(A^T H_1 + H_1 A)^{-1} c > 0.\) For \(\alpha = 1\), by assumption, the conditions of absolute stability are satisfied, \(-a^2 + b - c^2/4 > 0\), and for \(\alpha \to \pm \infty\) we

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