Optimal Pairs of Incomparable Clouds in Multisets

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Abstract. We consider the partially ordered set \((\mathbb{K}^n, \preceq)\), which is defined as \(n\)-th product of the chain \([k] = \{0, 1, 2, \ldots, k - 1\}\), and study pairs \((A, B)\) of incomparable sets \(A, B \subseteq \mathbb{K}^n\), that is, \(a \not\leq b, a \not\geq b\) for all \(a \in A, b \in B\) or (in short notation) \(A \nprec B\).

We are concerned with the growth of the functions \(f_n: \{0, 1, \ldots, k^n\} \to \{0, 1, \ldots, k^n\}, n \in \mathbb{N}\), defined by \(f_n(x) = \max\{|B|: A, B \subseteq \mathbb{K}^n \text{ with } |A| = x \text{ and } A \nprec B\}\) and a characterisation of pairs \((A, B)\), which assume this bound.

In the previously studied case \(k = 2\) our results are considerably sharper than earlier results by Seymour, Hilton, Ahlswede and Zhang.

1. Introduction, Basic Results and Problems

Let us be given the partially ordered set \(\mathcal{P}_n(\mathbb{K}^n, \preceq)\), where \([k] = \{0, 1, 2, \ldots, k - 1\}\) and \(a = (a_1, a_2, \ldots, a_n) \leq b = (b_1, b_2, \ldots, b_n)\) iff \(a_t \leq b_t\) for \(t = 1, 2, \ldots, n\).

In the terminology of our earlier work ([1], [8], [9]) we call a pair \((A, B)\) with \(A, B \subseteq [k]^n\) a cloud-antichain of length 2, if

\[ a \npreceq b, \quad a \npreceq b \quad \text{for all} \quad a \in A, \quad b \in B. \]  \hspace{1cm} (1.1)

A short expression for (1.1) is: \(A \npreceq B\).

We denote the set of these pairs by \(\mathcal{C}(n)\). The objects of our investigation are the functions \(f_n: \{0, 1, \ldots, k^n\} \to \{0, 1, \ldots, k^n\}, n \in \mathbb{N}\), defined by

\[ f_n(x) = \max\{|B|: \exists (A, B) \in \mathcal{C}(n) \text{ with } |A| = x\} \]  \hspace{1cm} (1.2)

and a characterization of pairs \((A, B)\) which are optimal, that is, assume this bound. We denote by \(\mathcal{O}(n)\) the set of all those optimal pairs.

In case where we emphasize the dependence on parameter \(k\) we also write \(\mathcal{C}(k)(n), \mathcal{C}_k(n), f_{n,k}(x)\), etc. instead of \(\mathcal{C}(n), \mathcal{O}(n), f_n(x)\), etc.

Previous work is discussed in [9], where the best results prior to those in this paper can be found. They are all for the binary alphabet, i.e. \(k = 2\). Familiarity with this paper may be helpful but is not necessary for an understanding of the present results and proofs. We extend here first the key result of that paper to the case of general \(k\).
Theorem 1. For every \( 0 \leq \gamma \leq k^{n-2} \), a pair \((A, B) \in \mathcal{C}(n)\) with \(|A| = \gamma\) exists, such that for some component all \(a\) in \(A\) have a 0 and for some other component all \(a\) in \(A\) have a \(k - 1\).

From here we derive by an approach similar to (but not identical with) that of [9] the main recursion.

Theorem 2. For every \( 0 \leq \gamma \leq k^{n-2} \),

(i) \( f_\gamma(y) = (k - 1)k^{n-1} + kf_{n-2}(y) - (k - 1)y \).

(ii) \( f_{s+2}(y) = (k^s - 1)(k^{n+s} - y) + k^sf_s(y) \) for \( s \geq 0 \).

The explicit characterization of all pairs \((\gamma, f_\gamma(y))\) given in [9] does not seem to allow a reasonably simple extension to general \(k\). Therefore results concerning aspects of this characterization problem are already of interest.

Theorem 8 of [1] states that in the case \(k = 2\) for \((A, B) \in \mathcal{C}(n)\)

(a) \(|A||B| \leq 2^{2^{n-4}}\)
(b) \(\min\{|A|, |B|\} \leq 2^{n-2}\)

and that these bounds are best possible.

The key observation was that for \((A, B) \in \mathcal{C}(n)\) we have the disjointness properties \((A \cap B) \cap (A \cup B) = \emptyset\), \((A \cap B) \cap (A \cup B) = \emptyset\), \((A \cap B) \cap (A \cup B) = \emptyset\), and \(A \cap B = \emptyset\).

Therefore

\(|A| + |B| + |A \lor B| + |A \land B| \leq 2^n\)

and since by the arithmetic-geometric means inequality

\[ (|A||B||A \lor B||A \land B|)^{1/4} \leq \frac{|A| + |B| + |A \lor B| + |A \land B|}{4} \]

we get

\[ |A||B||A \lor B||A \land B| \leq \left(\frac{2^n}{4}\right)^4. \tag{1.4} \]

Now we use the AD-inequality

\[ |A||B| \leq |A \lor B||A \land B| \] (see [6])

and get

\[ |A||B| \leq 2^{2^{n-4}}. \tag{1.5} \]

(b) is an immediate consequence.

Inspection shows that the same derivation is valid for all \(k\) and thus for \((A, B) \in \mathcal{C}(n)\)

\[ |A||B| \leq \left(\frac{k^n}{2}\right)^4 k^{2n-4}. \tag{1.6} \]

This is tight only for even \(k\).

In [9] the arithmetic-geometric means inequality was applied to two terms and so do we now for general \(k\). Hence,