ON THE NEWLY GENERALIZED ABSOLUTE CESÁRO SUMMABILITY OF ORTHOGONAL SERIES

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Dedicated to Professor Károly Tandori on the occasion of his 70th birthday
with admiration and friendship

1. Introduction. In [2] T. M. Flett defined a very useful extension of absolute Cesàro summability. According to his definition we shall say that a series $\sum a_n$ is summable $|C, \alpha, \gamma|_k$, where $k \geq 1$, $\alpha > -1$, $\gamma \geq 0$, if the series $\sum n^{\gamma k+k-1}|\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}|^k$ is convergent, $\sigma_n^{\alpha}$ being the $n$th Cesàro mean of order $\alpha$ of the series $\sum_n a_n$.

Among others, he proved the following result.

THEOREM. Let $r \geq k > 1$, $\gamma \geq 0$, $\alpha > \gamma - 1$, $\beta \geq \alpha + 1/k - 1/r$. Then if $\sum_{n=0}^{\infty} a_n$ is summable $|C, \alpha, \gamma|_k$ it is summable $|C, \beta, \gamma|_r$ and with $\tau_n^{\alpha} := n(\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha})$

$$\left\{ \sum n^{\gamma k+k-1}|\tau_n^{\beta}|^r \right\}^{1/r} \leq \left\{ \sum n^{\gamma k+k-1}|\tau_n^{\alpha}|^k \right\}^{1/k}.$$  

(1.1)

If $k = 1$, (1.1) holds when $r \geq 1$, $\gamma \geq 0$, $\alpha > \gamma - 1$, $\beta > \alpha + 1/k - 1/r$.

This theorem is a very important result, also in itself, moreover it has turned out that inequality (1.1) is crucial in the proofs of theorems concerning strong approximation of orthogonal series having approximation order $o_x(1/n^\gamma)$ (see e.g. G. Sunouchi [12], and [5], [6], [7]). Recently we intended to generalize that, this can be done, in our view and experience, only if previously we can generalize the Theorem by the same way, that is, if in the Theorem we can replace the factor $n^\gamma$ by a suitable factor $\gamma(n)$.

This was our motivation for generalizing this important result of Flett. Naturally, having found the method for such an extension, we used it for generalizing some further interesting theorems of Flett [2], see e.g. [8] and [9]. In [8] we introduced the newly generalized notion of absolute Cesàro summability, i.e. the definition of $|C, \alpha, \gamma(t)|_k$-summability, where $\gamma(t)$ is a positive nondecreasing function defined for $1 \leq t < \infty$. We say that the

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series $\sum_{n=0}^{\infty} a_n$ is summable $|C, \alpha, \gamma(t)|_k$ if the series
\[ \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k, \]
or briefly
\[ \sum_{n=1}^{\infty} \gamma(n)^k n^{-1}|\tau_n^\alpha|^k \]
is convergent.

Among others in [8] we proved the following theorems which will be used in the course of the proofs of our results to be presented in this work.

**Theorem A.** Let $r \geq k > 1$, $\alpha > -1$, $\beta \geq \alpha + 1/k - 1/r$, and $\gamma(t)$ be a non-decreasing positive function defined for $1 \leq t < \infty$ so that with some $C > 1$

\[ \limsup_{t \to \infty} \frac{\gamma(Ct)}{\gamma(t)} < C^{\alpha+1}. \]

If the series
\[ \sum_{n=0}^{\infty} a_n \]
is summable $|C, \alpha, \gamma(t)|_k$, then it is summable $|C, \beta, \gamma(t)|_r$ and

\[ \left\{ \sum \gamma(n)^r n^{-1}|\tau_n^\beta|^r \right\}^{1/r} \leq K \left\{ \sum \gamma(n)^k n^{-1}|\tau_n^\alpha|^k \right\}^{1/k}. \]

If $k = 1$, the result holds when $r \geq 1$, $\beta > \alpha + 1 - 1/r$ and (1.2) is satisfied.

If we keep $r = k$, then the factor $\gamma(n)$ on the left-hand side of (1.4) can be replaced by another factor $\mu(n)$ as follows.

**Theorem B.** Let $k \geq 1$, $\alpha > -1$, $\delta > 0$, $\beta \geq \alpha - \delta$, and $\beta > -1$, furthermore let $\mu(t)$ be a positive monotone, and $\gamma(t)$ a non-decreasing positive function defined for $1 \leq t < \infty$, so that

\[ C^\delta \limsup_{t \to \infty} \frac{\mu(Ct)}{\mu(t)} < \liminf_{t \to \infty} \frac{\gamma(Ct)}{\gamma(t)} \leq \limsup_{t \to \infty} \frac{\gamma(Ct)}{\gamma(t)} < C^{\alpha+1}. \]