CONVERGENCE RATES FOR PROBABILITIES OF MODERATE DEVIATION FOR SUMS OF RANDOM VARIABLES INDEXED BY $\mathbb{Z}_+^d$

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§ 1. Introduction

Let $\mathbb{Z}_+^d$, $d \geq 1$, be the positive integer $d$-dimensional lattice points with coordinate-wise partial ordering, $\leq$, i.e. for every $\overline{m} = (m_1, \ldots, m_d)$, $\overline{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$, $\overline{m} \leq \overline{n}$ if and only if $m_i \leq n_i$, $i = 1, \ldots, d$. Further, $|\overline{n}|$ is used for $\prod_{i=1}^{d} n_i$ and $|\overline{n}| \to \infty$ is interpreted as $n_i \to \infty$, $i = 1, 2, \ldots, d$.

Throughout the paper, $X$ and $\{X_{\overline{n}}, \overline{n} \in \mathbb{Z}_+^d\}$ are i.i.d. random variables and $S_\overline{n} = \sum_{k \leq \overline{n}} X_k, \alpha_1, \ldots, \alpha_d; \beta_1, \ldots, \beta_d; \gamma_1, \ldots, \gamma_d$ are real numbers with $0 < \alpha_i \leq 2, \beta_i \geq -1, i = 1, 2, \ldots, d$. Put

$$a(\overline{n}) = n_1^{\alpha_1} \cdots n_d^{\alpha_d}, \quad b(\overline{n}) = n_1^{\beta_1} \cdots n_d^{\beta_d}, \quad c(\overline{n}) = n_1^{\gamma_1} \cdots n_d^{\gamma_d}.$$  

Furthermore, put $\alpha = \max\{\alpha_i; 1 \leq i \leq d\}, s = \max\{\alpha_i(\beta_i + 1); 1 \leq i \leq d\}$, $r = \max\{\alpha_i(\beta_i + 2); 1 \leq i \leq d\}$, $q = \text{Card}\{\alpha_i(\beta_i + 2) = r\}$.

It is well-known that Gut [4] first studied the convergence of $P\left(|S_\overline{n}| \geq \varepsilon |\overline{n}|^{\frac{1}{\alpha}}\right)$ for $0 < \alpha < 2$. He showed that $\sum_{\overline{n}} |\overline{n}|^{\frac{r}{\alpha} - 2} P\left(|S_\overline{n}| \geq \varepsilon |\overline{n}|^{\frac{1}{\alpha}}\right) < \infty$ if and only if $E|X|^r (\log^+ |X|)^{d-1} < \infty$ and if $r \geq 1$, $EX = 0$. Klesov [6] extended the result of Gut to $a(\overline{n}) = n_1^{\alpha_1} \cdots n_d^{\alpha_d}, 0 < \alpha < 2$. Recently, the author further discussed the convergence rate of $P\left(|S_\overline{n}| \geq \varepsilon a(\overline{n})\right)$, and obtained some results.

For $\alpha = 2$, the sum cannot converge in view of the central limit theorem and for this case $|\overline{n}|^{\frac{1}{\alpha}}$ has to be replaced by $(|\overline{n}| \log |\overline{n}|)^{\frac{1}{2}}$, where $\log x = \max(1, \log x)$. (Similarly, $\log_2 x = \max(1, \log \log x)$ etc. Gut [5] has

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studied the convergence of \( P \left( |S_n| \geq \varepsilon \left( |n| \log |n| \right)^{\frac{1}{3}} \right) \) and has given necessary and sufficient conditions for the convergence of

\[
\sum_{n} |n|^{-\frac{2}{3}} P \left( |S_n| \geq \varepsilon \left( |n| \log |n| \right)^{\frac{1}{3}} \right)
\]

and

\[
\sum_{n} |n|^{-1} \log |n| P \left( |S_n| \geq \varepsilon \left( |n| \log |n| \right)^{\frac{1}{3}} \right)
\]

As above, if \( a(n) = n_{1}^{\alpha_{1}} \cdots n_{d}^{\alpha_{d}} \) and \( \alpha = 2 \), \( a(n) \) should be replaced by \( (a^{2}(n) \log a^{2}(n))^{\frac{1}{3}} \) and the convergence rates of

\[
P \left( |S_n| \geq \varepsilon \left( a^{2}(n) \log a^{2}(n) \right)^{\frac{1}{3}} \right)
\]

should also be studied. The main aim of this paper is to study these problems for \( \alpha = 2 \), thus generalizing the results of Gut [5]. In Section 3, we state some results, the proofs of which are given in Section 4. In Section 5 we give the corresponding theorems for \( P \left( |S_n| \geq \varepsilon \left( a^{2}(n) \log a^{2}(n) \right)^{\frac{1}{3}} \right) \) which give connections to the law of the iterated logarithm.

The main tool is due to Gut, whose approach is a direct estimation of the tail probabilities involved. But this approach cannot be applied in Section 5, so we have to use a different method, which is partly based on the approach of [1] and [2].

\[\text{§ 2. Auxiliary results}\]

In this section we give some lemmas needed later.

The sign \( \simeq \) is used to indicate that the quantities on either side converge simultaneously and the notation \( \approx \) between \( f(x) \) and \( g(x) \) to denote that there are constants \( C_1, C_2 \) such that for every \( x \) greater than some \( x_0 \),

\[
C_1 g(x) \leq f(x) \leq C_2 g(x).
\]

**Lemma 2.1.** Let \( \{\alpha_i, 1 \leq i \leq d\}, \{\tau_i, 1 \leq i \leq d\} \) and \( \{\gamma_i, 1 \leq i \leq d\} \) be real numbers with \( \alpha_i > 0 \) \((1 \leq i \leq d)\), \( \max \{\gamma_i, 1 \leq i \leq d\} < 0 \) or \( \min \{\gamma_i, 1 \leq i \leq d\} > 0 \), and \( a(n) = n_{1}^{\alpha_{1}} \cdots n_{d}^{\alpha_{d}}, t(n) = n_{1}^{\gamma_{1}} \cdots n_{d}^{\gamma_{d}} \), \( c(n) = n_{1}^{\nu_{1}} \cdots n_{d}^{\nu_{d}} \). Put \( r = \max \{\alpha_i(\tau_i + 1), 1 \leq i \leq d\} \), \( q = \text{Card} \{ i, \alpha_i(\tau_i + 1) \leq 0 \} \).