Abstract

We are going to investigate simultaneous extensions of various topological structures (i.e. traces on several subsets at the same time are prescribed), also with separation axioms $T_0$, $T_1$, symmetry (in the sense of Part I, § 3), Riesz property, Lodato property. The following questions will be considered: (i) Under what conditions is there an extension? (ii) How can the finest extension be described? (iii) Is there a coarsest extension? (iv) Can we say more about extensions of two structures than in the general case? (v) Assume that certain subfamilies (e.g. the finite ones) can be extended; does the whole family have an extension, too? The general categorial results from Part I will be applied whenever possible (even they are not really needed).

§ 16 Riesz-type structures

16.1 In this section, we consider extensions of Riesz-type structures other than proximities. The results are similar to those for Riesz proximities (see §11), but the degree of similarity varies with the structures.

A merotopy $\mathcal{M}$ on $X$ is Riesz if

\[ \text{int} \, c \in \text{Cov} \, X \ (c \in \mathcal{M}), \]

where $\text{int} \, c = \{ \text{int} \, C : C \in c \}$, $\text{int} = \text{int}_c$, $c = c(\mathcal{M})$.

Equivalently, with $\nu \mathcal{M}$ denoting the collection of the far systems (i.e. $\nu \mathcal{M} = \exp \exp \ X \setminus \nu \mathcal{M}$):

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\( \nu \text{MR.} \) if \( a \in \nu \mathcal{M} \) then \( \bigcap c_a = \emptyset \);
\( \mu \text{MR.} \) \( n(x) \in \mu \mathcal{M} \) \( (x \in X) \).

Here \( ca = \{cA : A \in a\} \) and \( n = n_c \). A micromeric filter is called a \textit{Cauchy filter}. Thus \( \mu \text{MR} \) states that each neighbourhood filter is Cauchy. If \( \mathcal{M} \) is Riesz then \( c(\mathcal{M}) \) is weakly topological.

Most of the contents of § 11 can be repeated for Riesz merotopies, replacing near (or far) pairs of sets by near (or far) systems, and compressed filters by Cauchy filters. Let us first note that the Riesz merotopies form a strongly reflective subcategory \( \text{RMer of Mer} \), hence \( \text{RMer} \) is an extension category, and the formulas 8.2 (1) to (6) (and their versions with \( \nu \mathcal{M} \) and \( \mu \mathcal{M} \)) remain valid, except that \( \inf_R \neq \inf = \inf_H \) and \( \sup_R = \sup \neq \sup_H \).

The Riesz reflexion \( \mathcal{M}_R \) of \( \mathcal{M} \) can be easily described, on the model of Lemma 11.1:

(1) \( a \in \nu \mathcal{M}_R \) iff \( a \in \nu \mathcal{M} \) or \( \bigcap c_R a \neq \emptyset \).

Moreover, \( c(\mathcal{M}_R) = c_R \). With covers:

(2) \( c \in \mathcal{M}_R \) iff \( c \in \mathcal{M} \) and \( \text{int}_R c \in \text{Cov}_X \),

where \( \text{int}_R = \text{int}_{c_R} \).

16.2 Let us be given a family of Riesz merotopies. By [3] Theorem 3.2, there is a Riesz extension compatible with a given symmetric weakly topological extension of the induced closures \( c_i \) iff the trace filters are Cauchy. Replacing \( c \) by a finer extension of \( c_i \), the trace filters get finer, i.e., they remain Cauchy. So if there is a Riesz extension then there is one compatible with \( c_R \). Similarly to 11.5, the condition that the trace filters are Cauchy can be described in terms of far systems. So we have:

PROPOSITION. A family of Riesz merotopies has Riesz extensions iff the following holds for each \( i \in I \):

(1) \( a_i \in \nu \mathcal{M}_i \) implies \( \bigcap c_R a_i = \emptyset \);

if so then \( \mathcal{M}_R \) is the finest Riesz extension. 

\( \mathcal{M}_R \) is not always an extension (not even for \( T_1 \) Riesz merotopies): in Example 11.3, replace the Euclidean proximities by Euclidean merotopies. Similarly to 11.6, if a family of \( T_1 \) merotopies has Riesz extensions then \( \mathcal{M}_R \) is \( T_1 \).

16.3 PROPOSITION. A family of two Riesz merotopies always has Riesz extensions.

PROOF. Just as the proof of Proposition 11.7, with Cauchy instead of compressed.

There is in general neither a coarsest Riesz extension nor a coarsest \( T_1 \) Riesz extension: in Example 11.6, replace \( \delta_i \) by \( \mathcal{M}^0(\delta_i) \). Differently from the case of