THE FINITE ELEMENT METHOD OF SINGULAR PERTURBATION PROBLEM

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ABSTRACT

In this paper we construct new finite element subspace using polynomials of different degrees and the new finite element scheme is established. The convergence of the scheme and the stability of the reduced difference equation are proved.

I. Problem of Differential Equation

We consider the following problem of the differential equation in the region $R+\Gamma$: \(0 \leq x, y \leq 1\)

\[
\begin{aligned}
&\epsilon \left( -\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right) - a_1 \frac{\partial u}{\partial x} - a_2 \frac{\partial u}{\partial y} - a_3 u = f \quad (x, y) \in R \\
&u|_{r} = 0
\end{aligned}
\]  

(1.1)

where $\Gamma$ is the boundary of $R$, $\epsilon$ is a small parameter satisfying $0 < \epsilon \ll 1$, $a_1$, $a_2$, $a_3$ are positive constants, $f$ is a given function.

When $\epsilon = 0$, the problem (1.1) is degenerated into the following problem:

\[
\begin{aligned}
&- a_1 \frac{\partial u}{\partial x} - a_2 \frac{\partial u}{\partial y} - a_3 u = f \\
&u|_{r} = 0
\end{aligned}
\]  

(1.2)

where $\Gamma_r = \Gamma \cap \{x = 0, y = 0\}$.

II. The Construction of the Subspace and the Establishment of Finite Element Scheme

Define the continuous bilinear and linear forms as follows:

\[
\begin{aligned}
\sigma (v, w) &= \iint_{S_1} \left( -\epsilon v_w w_x - \epsilon v_y w_y - a_1 v_x w - a_2 v_y w - a_3 v w \right) dx dy \quad \forall v, w \in V \\
f(v) &= \iint_{S_1} f v dx dy \quad \forall v \in V
\end{aligned}
\]  

(2.1)

And the variational formulation of (1.1) is then:
Find \( u \in V \) such that
\[
o(u, v) = f(v) \quad \forall v \in V
\] (2.2)

Take the grid spacings \( h, r \), and the natural numbers \( M \) and \( N \) such that \( Mh = 1 \) and \( Nr = 1 \) respectively.

Let
\[
I_i = [x_{i-1}, x_i], \quad J_i = [y_{i-1}, y_i], \quad K_{ij} = I_i \times J_j \quad (i=1(1)M, j=1(1)N)
\]

We introduce the parameters \( \theta_1, \theta_2 \) satisfying:
\[
\lim_{s \to 0^+} \frac{\theta_1 h}{2s} = 1, \quad \lim_{s \to 0^+} \frac{\theta_2 r}{2s} = 1 \quad (2.3)
\]

Let
\[
I_i = [x_{i-1}, x_{i-\theta_1}], \quad J_i = [x_{i-\theta_1}, x_i], \quad J_i = [y_{i-1}, y_i], \quad J_i = [y_{i-1}, y_i]
\]

Then
\[
K_{ij} = K_i \cup K_i \cup K_i \cup K_i
\]

where
\[
K_i = I_i \times J_i, \quad K_i = I_i \times J_i, \quad K_i = I_i \times J_i, \quad K_i = I_i \times J_i
\]

we construct a finite element subspace \( V^h \subset V \) in the following manner:
\[
V^h = \begin{cases}
V^h_{K_i} \in Q_{h,h}, & V^h_{K_i} \in Q_{h,h}, \\
V^h_{K_i} \in Q_{h,h}, & V^h_{K_i} \in Q_{h,h}, \\
0 & (i=1(1)M, j=1(1)N)
\end{cases}
\] (2.4)

Here \( Q_{h,h} \) denotes the space of polynomials in the two variables \( x, y \) of degree \( h \) in \( x \) and \( l \) in \( y \).

The basis functions \( \varphi_{ij} \) are given by:
\[
\varphi_{ij} = \begin{cases}
\frac{(x-x_{i-\theta_1})(y-y_{j-\theta_1})}{(x_i-x_{i-\theta_1})(y_j-y_{j-\theta_1})} & (x, y) \in K_i
\\
\frac{(x-x_{i-1})(y-y_{j-1})}{(x_{i+1}-x_{i})(y_j-y_{j-1})} & (x, y) \in K_i
\\
\frac{(x-x_{i-1})(y-y_{j-1})}{(x_{i+1}-x_{i})(y_j-y_{j-1})} & (x, y) \in K_i
\\
\frac{(x-x_{i-1})(y-y_{j-1})}{(x_{i+1}-x_{i})(y_j-y_{j-1})} & (x, y) \in K_i
\\
\frac{(x-x_{i-1})(y-y_{j-1})}{(x_{i+1}-x_{i})(y_j-y_{j-1})} & (x, y) \in K_i
\\
\frac{(x-x_{i-1})(y-y_{j-1})}{(x_{i+1}-x_{i})(y_j-y_{j-1})} & (x, y) \in K_i
\\
\frac{(x-x_{i-1})(y-y_{j-1})}{(x_{i+1}-x_{i})(y_j-y_{j-1})} & (x, y) \in K_i
\\
\frac{(x-x_{i-1})(y-y_{j-1})}{(x_{i+1}-x_{i})(y_j-y_{j-1})} & (x, y) \in K_i
\\
\end{cases}
\] (2.5)

And the discrete formulation of (2.2) is then taken as: