PARTITIONS OF VECTOR SPACES

P. KOMJÁTH (Budapest)

0. Introduction

Let $V$ be a vector space over $\mathbb{Q}$, the rationals, and consider

\[(0.1) \quad \sigma : \sum_{j=1}^{n} \lambda_{ij}x_j = 0 \quad (1 \leq i \leq m),\]

a system of homogeneous linear equations over $\mathbb{Q}$. Is it true, that if $V$ is colored with countably many colors (is $\omega$-colored, in short), then there is a monocolored solution of $\sigma$ satisfying

\[(0.2) \quad x_j \neq x_{j'} \quad (j \neq j').\]

Clearly, the answer depends on the dimension of $V$ which is the same as the cardinal of $V$ assuming $V$ is uncountable. (For countable $V$, the answer is obviously "no".) By a theorem of Erdős–Kakutani [5], every vector space of size $\leq \omega_1$ is the union of countably many linearly independent sets, so the dimension in question must be at least $\omega_2$. Let $\lambda(\sigma)$ be the least dimension such that the answer to our question is "yes". If no such cardinal exists, we write $\lambda(\sigma) = \infty$. An old observation of Rado’s (?) is that $\lambda(x + y = 2z) = \infty$. P. Erdős showed many years ago, that $\lambda(x + y = z + t) = \omega_2$. The question, therefore, arises, how to determine $\lambda(\sigma)$, or at least, which cardinals occur as the values of $\lambda$. As there are only countably many systems of linear equations, clearly there is a cardinal bigger than every $\lambda(\sigma)$ with $\lambda(\sigma) < \infty$. In Theorem 1.3, we prove that the supremum of $2^\omega, 2^{2^\omega}, \ldots$ is such a bound. This statement is a special case of a two-cardinal theorem of Vaught (see [2], [8]), but we think that our proof is simpler in this case. It is not clear, if that is the least upper bound, we have only been able to find an example with $\lambda(\sigma) = (2^\omega)^+$. It is, however, the least upper bound, if GCH is assumed, as we show that every $\omega_n(2 \leq n < \infty)$ is $\lambda(\sigma)$ for some $\sigma$. We further show that every system $\sigma$ with

\[\text{Research supported by OTKA grant No. T014105}\]
\[\text{Mathematics subject classification numbers, 1991. Primary 03E05, 04A20, 05D10.}\]
\[\text{Key words and phrases. Ramsey problems, linear equations.}\]
\( \lambda(\sigma) \leq 2^\omega \) satisfies a certain property which in turn implies that \( \lambda(\sigma) \leq R_\omega \). An interesting corollary of this result is that if \( \lambda(\sigma) \leq 2^\omega \), then \( \lambda(\sigma) \leq R_\omega \). This also follows from S. Shelah’s two-cardinal theorem [7].

**Notation.** We use the standard set theory notation. The well-ordering theorem is assumed. Cardinals are identified with initial ordinals. If \( \kappa = R_\alpha \), then \( \kappa^+ = R_{\alpha+1} \). \( 2^\omega \) is the cardinal of the continuum.

1. **A bound for \( \lambda(\sigma) \)**

**Theorem 1.1.** \( \lambda(x + y = z) = (2^\omega)^+ \).

**Proof.** To show that \( \lambda(x + y = z) > 2^\omega \), it suffices to give a decomposition of \( \mathbb{R} \) into countably many pieces with no solution of the above mentioned equation. Decompose in such a way that the ratio of two numbers in the same class should always be between 1 and 2.

For the other direction, let \( B = \{ b_\alpha : \alpha < (2^\omega)^+ \} \) be a basis in a vector space of dimension \( (2^\omega)^+ \). If we color the differences \( \{ b_\alpha - b_\beta : \alpha < \beta < (2^\omega)^+ \} \) with countably many colors, then, by the Erdős–Rado theorem [6], there exist \( \alpha < \beta < \gamma \) such that \( b_\alpha - b_\beta, b_\beta - b_\gamma \) and \( b_\alpha - b_\gamma \) get the same color, thereby giving a monocolored solution of our equation.

**Definition 1.2.** \( \exp_0(\omega) = R_0, \exp_{n+1}(\omega) = 2^{\exp_n(\omega)} \) and let the cardinal \( \exp_\omega(\omega) \) be the supremum of the cardinals \( \exp_n(\omega) \) \((n < \omega)\).

**Theorem 1.3.** If \( \lambda(\sigma) < \infty \) for some system \( \sigma \) of equations, then \( \lambda(\sigma) < \exp_\omega(\omega) \).

**Proof.** Assume that there is a vector space, \( V \) with basis \( \{ b_\alpha : \alpha < \kappa \} \) witnessing \( \lambda(\sigma) < \omega \). Color the nonzero vectors of \( V \) as follows. If

\[
(1.1) \quad v = \sum_{i=1}^{k} \gamma_i b_{\alpha_i}, \quad (\alpha_1 < \cdots < \alpha_k)
\]

then assign the ordered sequence of rationals \( (\gamma_1, \ldots, \gamma_k) \) to \( v \). Give a separate color to 0. Clearly, this is a coloring with countably many colors. By our assumptions, there are \( n \) different vectors, getting the same color, which form a solution of \( \sigma \). This means, that there are \( K < \omega \) ordinals, \( \alpha_1 < \cdots < \alpha_K < \kappa \) such that among the vectors which can be written in the form

\[
(1.2) \quad \sum_{i=1}^{k} \gamma_i b_{\alpha_{j_i}} \quad \text{with} \quad 1 \leq j_1 < \cdots < j_k \leq K,
\]

there are some \( n \) which solve \( \sigma \). Now assume that a vector space with basis \( \{ b_\alpha : \alpha < (\exp_k(\omega))^+ \} \) is colored with countably many