SPACES BETWEEN X AND ITS FREUDENTHAL COMPACTIFICATION

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Abstract

Let $\phi X$ indicate the Freudenthal compactification of a rimcompact, completely regular Hausdorff space $X$. In this paper the spaces $Y$ which satisfy $X \subseteq Y \subseteq \phi X$ are characterized. From this a characterization of when $X$ lies between its locally compact part $L(X)$ and $\phi(L(X))$ follows. Such spaces necessarily possess a compactification $\alpha X$ for which $\text{Cl}_{\alpha X}(\alpha X - X)$ is 0-dimensional. Conditions, including those internal to $X$, are provided which are necessary and sufficient for this property to hold.

1. Introduction

In this paper all spaces are completely regular and Hausdorff. A space $X$ is rimcompact if each point of $X$ has a base of open sets with compact boundaries. The Freudenthal compactification $\phi X$ of a rimcompact space $X$ is maximal with respect to the property that $\phi X - X$ is 0-dimensional. (See [1, 5, 6].) The aim of this paper is to characterize the spaces $Y$ which lie between $X$ and $\phi X$, that is, $Y$ satisfies $X \subseteq Y \subseteq \phi X$. Characterizations for when $Y$ lies between $X$ and $\beta X$, the Stone–Čech compactification of $X$, are abundant and may be found in [4], for example.

Also, suppose $X$ is any space for which the locally compact part $L(X)$ of $X$ is dense and the set $R(X) = X - L(X)$ of points not having compact neighborhoods is 0-dimensional. A characterization is obtained for when $L(X) \subseteq X \subseteq \phi(L(X))$ holds. Such a space must possess a compactification $\alpha X$ for which the condition that $\text{Cl}_{\alpha X}(\alpha X - X)$ be 0-dimensional holds. Conditions, including those internal to $X$, are given which are necessary and sufficient for this property to hold. An example is provided of a space $X$ for which some $\text{Cl}_{\alpha X}(\alpha X - X)$ is 0-dimensional but where $X$ is not a subspace of $\phi(L(X))$.


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2. The main result

An open set $U$ in a space is said to be $\pi$-open if the boundary, $Fr_X U$, of $U$ is compact. (See [1].) Recall that a compactification $\alpha X$ of $X$ is perfect if and only if whenever a closed set $F$ separates $X$ into disjoint non-empty open sets $H_1$ and $H_2$, then $Cl_{\alpha X} F$ separates $\alpha X$ into disjoint open $G_1$ and $G_2$ where $G_i \cap X = H_i$, for $i = 1, 2$. The Freudenthal compactification $\phi X$ is the minimal perfect compactification of $X$. (See [6] and [8].) This provides motivation for the following:

**Definition 2.1.** We say that $X$ is a perfect subspace of $Y$ if and only if $X$ is dense in $Y$ and whenever $U$ is a $\pi$-open set in $X$, then there is a $\pi$-open set $U_1$ in $Y$ satisfying $U = U_1 \cap X$ and $Fr_Y U_1 = Fr_X U$.

We will need the following result in the proof of the main theorem.

**Lemma 2.2.** Suppose $X$ is a rimcompact perfect subspace of $Y$. If every point of $Y - X$ has a base of $\pi$-open $Y$-neighborhoods with boundaries in $X$, then $Y$ is rimcompact.

**Proof.** We need only to consider points in $X$. Let $O_p$ be any $Y$-open neighborhood of a point $p \in X$. Choose a $Y$-open neighborhood $N_p$ of $p$ such that $Cl_Y N_p \subseteq O_p$. Now $N_p \cap X$ contains a $\pi$-open $X$-neighborhood $U_p$ of $p$. By assumption, there is a $\pi$-open set $V_p$ in $Y$ such that $U_p = V_p \cap X$ and $Fr_Y V_p = Fr_X U_p$. Then $V_p \subseteq Cl_Y V_p = Cl_Y (V_p \cap X) \subseteq Cl_Y N_p \subseteq O_p$. Thus, $V_p \subseteq O_p$ so that $p$ has a base of $\pi$-open neighborhoods in $Y$ and the proof is complete. $\blacksquare$

In what follows a topological space is considered to be identified with any homeomorphic image of it.

**Theorem 2.3.** Let $X$ be a rimcompact, dense subspace of $Y$. Then $Y$ satisfies $X \subseteq Y \subseteq \phi X$ if and only if $X$ is a perfect subspace of $Y$ and each point of $Y - X$ has a base of $\pi$-open $Y$-neighborhoods with boundaries in $X$. Moreover, under these conditions $\phi X = \phi Y$.

**Proof.** Suppose $X \subseteq Y \subseteq \phi X$. Then each point of $\phi X$ has a base of neighborhoods with compact boundaries which lie in $X$. (See VI.30 of [5].) Thus, all points of $Y - X$ also possess this property.

Next let $U$ be any $\pi$-open set in $X$. If $Cl_X U = X$, set $U_1 = Y - Fr_X U$. Otherwise, since $\phi X$ is perfect, $Fr_X U$ separates $\phi X$ into disjoint $V$ and $W$ such that $V \cap X = U$ and $W \cap X = X - Cl_X U$. Evidently, $Fr_{\phi X} V = Fr_X U$. Setting $U_1 = V \cap Y$ provides the required $\pi$-open subset of $Y$. Hence $X$ is a perfect subspace of $Y$.

Conversely, assume the conditions on $Y$. By Lemma 2.2, $Y$ is rimcompact so that $\phi Y$ exists. The inclusion map of $X$ into $Y$ has an extension $f$ mapping $\beta X$ onto $\beta Y$. Let $p_1$ and $p_2$ be the natural projections of $\beta X$ and $\beta Y$ onto $\phi X$ and $\phi Y$. 