SPACES FROM PIECES I

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Abstract

Consider the following general problem: let topological structures $s_i$ be given on subsets $X_i$ of a fixed set $X$; under what conditions does there exist a structure $s$ on $X$ such that $s|X_i = s_i$ for each $i$? In the present paper, the problem will be investigated in topological categories satisfying some simple additional conditions; in Parts II to IV, we shall deal with some specific structures.

Let $X$ be a set, $X_i \subseteq X$ ($i \in I$), and let us be given a topology $c_i$ on each $X_i$. Now is there a topology $c$ on $X$ such that each $(X_i, c_i)$ is a subspace of $(X, c)$? (It is, of course, necessary to assume that the topologies coincide on the intersections.) The answer to this question belongs to the folklore: yes for two topologies, no for more than two. It is the aim of this series (consisting of four parts) to investigate the analogous problem for several kinds of topological structures. The answer depends very much on what structures we consider: e.g. two regular topologies may not have an extension; on the other hand, an arbitrary family of merotopies has a simultaneous extension in the above sense. Nevertheless, there are some results that hold in any topological category satisfying some additional conditions (fulfilled in all the important categories); more can be said under a strong assumption (which does not hold e.g. in Top).

The present paper deals with the general categorical problem. Specific structures will be considered in Parts II to IV. See [2–5] for the following related problem: given a structure (e.g. a closure) on $X$, and compatible richer structures (e.g. merotopies) on each $X_i$, is there a compatible simultaneous extension?

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§ 1. Simultaneous extensions in topological categories

1.1 A topological category over \( \text{Set} \) carries much information that is irrelevant from the point of view of extensions: it is not necessary to know which of the non-injective maps are continuous. Therefore we shall consider topological categories over \( \text{mSet} \), the object-full subcategory of \( \text{Set} \) with all the injective maps as morphisms (\( m \) stands for monomorphism). If \( C \) is a concrete category over \( \text{Set} \) then \( mC \) denotes the object-full subcategory of \( C \), considered as a concrete category over \( \text{mSet} \), in which a map is continuous iff it is injective and continuous in \( C \). (We shall always speak about continuous maps instead of morphisms, and a structured set will be called a space rather than an object.)

Let \( C \) be a \textit{topological category over} \( \text{mSet} \); it is included in the definition that the initial structures are unique (this assumption is convenient, but not really necessary). For sets \( X_0 \subset X \), \( m_{X_0,X} \) denotes the inclusion map from \( X_0 \) into \( X \). Given structures \( s \) on \( X \), \( t \) on \( Y \), and an injection \( f : X \to Y \), \( f^{-1}t \) is the initial structure, and \( fs \) is the final structure with respect to \( f \). (In topological categories over \( \text{Set} \), the same notation will be used with arbitrary maps \( f \).) If \( X_0 \subset X \), \( s_0 \) is a structure on \( X_0 \) and \( s \) on \( X \) then \( s|X_0 = m_{X_0,X}^{-1}s \) is the \textit{restriction} of \( s \) to \( X_0 \) (thus \( (X_0, s|X_0) \) is a \textit{subspace} of \( (X, s) \) in the usual sense), and \( s_0|X = m_{X_0,X}s_0 \) is the \textit{prolongation} of \( s_0 \) to \( X \). (The same notation can be used, because \( m_{X,X}^{-1}s = m_{X,X}s = s \).)

The following notations will be used in a topological category over \( \text{Set} \) or \( \text{mSet} \): \( s < t \) means that \( s \) is \textit{coarser} than \( t \) (\( t \) is \textit{finer} than \( s \)), i.e. \( m_{X,X} \) is \( (t, s) \)-continuous; \( \inf (= \text{infimum}) \) and \( \sup (= \text{supremum}) \) are to be understood with respect to this partial order. (The notation \( t \leq s \) preferred by categorical topologists is clearly incompatible with the usual topological meaning of infimum and supremum.)

1.2 A topological category over \( \text{mSet} \) can be defined by giving the structures, the relation finer/coarser, and the restrictions. The following conditions have to be satisfied:

a) On each set, the structures form a non-empty complete lattice (possibly a proper class).

b) If \( s \) is a structure on \( X \) then \( s|X = s \).

c) If \( s \) is a structure on \( X \) and \( X_1 \subset X_0 \subset X \) then \( s|X_1 = (s|X_0)|X_1 \).

d) The supremum commutes with the restriction.

e) \( < \) and the restriction are defined in the same way on equivalent sets.

More precisely, e) means that:

\( e_1 \) if \( f : X \to Y \) is a bijection then there is a bijection \( \bar{f} \) between the structures on \( X \) and the structures on \( Y \) such that \( \bar{f}(s) < \bar{f}(t) \) iff \( s < t \);

\( e_2 \) if \( X_0 \subset X \), \( f_{X_0} : X_0 \to Y_0 = f[X_0] \) is defined by \( f_{X_0}(x) = f(x) \) (\( x \in X_0 \)), and \( s \) is a structure on \( X \) then \( f(s)|Y_0 = f_{X_0}(s|X_0) \).

Now an injection \( f : X \to Y \) is \( (s, t) \)-continuous iff \( t|f[X] < \bar{f}(s) \).

1.3 Let us be given a topological category \( T \) over \( \text{Set} \). An initial structure in \( T \) belonging to a source with injective maps is clearly initial in \( mT \), too; conversely, if a structure is initial in \( mT \) then it is the coarsest structure for which the injections in the source are continuous, hence it is the initial structure in \( T \) belonging to the