A Method for Solving a Quadratic Optimal Control Problem

RUSSELL D. RUPP

Communicated by M. R. Hestenes

Abstract. Hestenes' method of multipliers is used to approximate a quadratic optimal control problem. The global existence of a family of unconstrained problems is established. Given an initial estimate of the Lagrange multipliers, a convergent sequence of arcs is generated. They are minimizing with respect to members of the above family, and their limit is the solution to the original differentially constrained problem.

1. Introduction

The problem considered here is to minimize a quadratic functional over a class of arcs which satisfy a system of linear differential equations. The technique of minimization is Hestenes' method of multipliers (Ref. 1). The global existence of a family of unconstrained problems which is parameterized by Lagrange multipliers is first established. This family contains the original problem when the appropriate set of multipliers is considered. Given an initial estimate of the Lagrange multipliers, a sequence of arcs which converges to the original constrained minimizing arc is generated. Each arc in this minimizing sequence is the solution to a problem in the above family. This method has the advantage that there is no singularity as the approximating solutions converge. For this reason, it may be preferable to a penalty function method.

An arc $x$ is a collection of constants and real-valued functions:

$$x : b^x, x^o(t), u^o(t).$$

---

1 Paper received April 30, 1970. The preparation of this paper was sponsored in part by the U.S. Army Research Office under Grant No. DA-31-124-ARO(D)-355.

2 Formerly, Graduate Student, Department of Mathematics, University of California at Los Angeles, Los Angeles, California. Presently, Assistant Professor, Department of Mathematics, State University of New York, Albany, New York.

The ranges of $h$, $i$, $k$ are always $1 \leq h \leq r$, $1 \leq i \leq n$, $1 \leq k \leq m$, unless otherwise indicated. The functions $x^i(t)$, $u^k(t)$ have domain $T^1 \leq t \leq T^2$, where $T^1 < T^2$ are constants. The functions $x^i(t)$ are absolutely continuous, and their derivatives $\dot{x}^i(t)$, along with the functions $u^k(t)$, are square integrable. We often write the arc $x$ in the vector form

$$x : b, x(t), u(t),$$

where $b$, $x(t)$, $u(t)$ are vectors with components given by $b^h$, $x^i(t)$, $u^k(t)$. An arc $x : b, x(t), u(t)$ is called terminally admissible and said to belong to the class $\mathcal{U}$ if the linear functional end conditions

$$x^i(t_s) = a^i_s b^h + a^i_s g$$

hold, where $a^i_s$ and $a^i_s$ are constants. The index $s$ always has the range $1 \leq s \leq 2$. Here and later on, repeated indices denote summation with respect to that index. An arc $x : b, x(t), u(t)$ is called differentially admissible if the differential constraints

$$\dot{x}(t) = a^h b^h + c^i(t) x^i(t) + d^k(t) u^k(t)$$

hold almost everywhere in $T^1 \leq t \leq T^2$. Here, the functions $a^i(t)$, $c^i(t)$ are square integrable, and the functions $d^k(t)$ are essentially bounded. Finally, the arc $x : b, x(t), u(t)$ is said to belong to a $btx_{x/u}$-neighborhood $\mathcal{N}$ if $(b, t, x(t), \dot{x}(t), u(t))$ is in $\mathcal{N}$ for almost all $T^1 \leq t \leq T^2$. Similar convention is made with respect to the other combinations of variables.

By the statement that a functional is quadratic or of degree two, we mean that it may be written as the sum of a quadratic form plus a linear term plus a constant. Similarly, by a linear functional or a function of degree one, we mean a function which is linear plus a constant.

The problem considered here is to minimize a functional $I(x)$, which is of degree two, over the class of terminally and differentially admissible arcs. Thus, one has

$$I(x) = \frac{1}{2} Q_1(x) + L(x) + Q_0,$$

where

$$Q_1(x) = \int_{T^1}^{T^2} \left[ Q^{h,i}(t) b^h b^i \right. + Q^{i,i}(t) x^i(t) x^i(t) \left. \right] dt,$$

$$L(x) = \int_{T^1}^{T^2} \left[ L^{h,k}(t) b^h + L^{i,i}(t) x^i(t) \right] dt.$$