SEPARATION AND APPROXIMATION THEOREMS
ON DERIVATIVES

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1. Introduction

In this paper, we consider the following problems:
(i) (the problem of uniform approximation) characterize the sets \( H \subset [0, 1] \)
having the property that, for every Baire 1 function \( f(x) \) and for every \( \varepsilon > 0 \), there
exists a derivative \( g(x) \) such that
\[
|f(x) - g(x)| < \varepsilon \quad (x \in H);
\]
(ii) (the problem of separation) characterize the pairs \( H_1, H_2 \) \((H_1 \subset [0, 1],
H_2 \subset [0, 1])\) for which there exists a derivative \( g(x) \) with \( g|_{H_1} = 0, g|_{H_2} = 1 \).

These two problems are closely related to each other. We are going to prove
here a separation theorem (certainly not the best possible) by the aid of which we
give a full solution of the approximation problem.*

As for (i) two instances were proved in [1]:
(ia) if \( H \) is a nowhere dense closed set, then the uniform approximation property
holds on \( H \) ([1], Corollary 4.9);
(ib) if \( H \) is a set of measure zero, then, for every Baire 1 function \( f(x) \), there
exists a derivative \( g(x) \) with \( f(x) = g(x) \) \((x \in H)\) ([1], theorem 4.14).

As for (ii) an important special case was proved by Z. Zahorski (as a matter
of fact, it concerns to approximately continuous functions, see [3], Lemma 12):
(iiia) if \( H_1, H_2 \) are disjoint \( G_\delta \) and \( D \)-closed sets, then there exists an approxi-
mately continuous function \( f(x) \) with \( f|_{H_1} = 0, f|_{H_2} = 1 \), and \( 0 < f(x) < 1 \) elsewhere.

2. Notations and lemmas

We apply here the same notations as in [1] and we need some results proved
in [1]. For the sake of the reader’s convenience we repeat here the notations and
list the referred results as lemmas.
— Let \( \mathcal{B} \) and \( \mathcal{D} \) denote the family of Baire 1 functions and derivatives, re-
spectively (all these functions are defined on \([0, 1]\)).
— If \( \mathcal{F} \) is any family of functions and \( H \subset [0, 1] \), then \( \mathcal{F}_H \) denotes the family
of restricted \( \mathcal{F} \)-functions, i.e. \( \mathcal{F}_H = \{ f|_H : f \in \mathcal{F} \} \).
— \( \lambda(H) \) denotes the Lebesgue outer measure of \( H \).
— The set \( H \) is called \( D \)-open, if every \( x \in H \) is of inner density 1 in \( H \).

* The complete solution of the separation problem has recently been found by M. Laczko-
vich ([2]).
A measurable set $H$ is said to be metrically dense in an interval $[a, b]$ if, for every subinterval $I \subseteq [a, b]$, $\lambda(I \cap H) > 0$.

The closure and the interior of any set $H$ is denoted by $\text{cl} \ H$ and $\text{int} \ H$, respectively.

**Lemma 1** (see [1], Lemma 4.3). For any subset $\emptyset \neq E \subseteq (-\infty, +\infty)$ we define

$$h(x, E) = \inf \{|x-y|: y \in E\}^2.$$  

Then $h(x, E)$ is continuous and for $x \in \text{cl} \ E$ even the derivative $h'(x, E)$ exists and

$$h(x, E) = h'(x, E) = 0 \quad (x \in \text{cl} \ E).$$

If further $F(x)$ is differentiable then, for any $G(x)$ with $F(x) \leq G(x) \leq F(x) + h(x, E)$, the relations

$$G'(x) = F'(x) \quad (x \in \text{cl} \ E), \quad G(x) = F(x) \quad (x \in \text{cl} \ E)$$

hold (the proof is trivial).

**Lemma 2** (see [1], Corollary 4.7). Let $E_1, \ldots, E_n, \ldots$ and $E = \bigcup E_n$ be nowhere dense and closed sets, $E_i \cap E_j = \emptyset$ ($i \neq j$). Let $f(x)$ be defined on $E$ and suppose that $f|_{E_n}$ is constant for $n=1, 2, \ldots$. Then $f \in \mathcal{O}_E$.

**Lemma 3.** If $H \subseteq (a, b)$; $(a, b) \setminus H$ is metrically dense in $(a, b)$, $\varphi(x) \leq \psi(x)$ are continuous functions in $(a, b)$, then there exist differentiable functions $F(x)$ and $G(x)$ such that

(i) $F'(x) = 0 \quad (x \in H)$ and $\varphi(x) \leq F(x) < \psi(x) \quad (a < x < b)$,

(ii) $G'(x) = 1 \quad (x \in H)$ and $\varphi(x) < G(x) \leq \psi(x) \quad (a < x < b)$.

**Proof.** Assertion (i) is Corollary 4.12 in [1]. In order to prove (ii) we put $\varphi_1(x) = \varphi(x) - x$, $\psi_1(x) = \psi(x) - x$ and applying (i) we find $F(x)$ such that $F'(x) = 0 \ (x \in H)$ and

$$\varphi_1(x) \leq F(x) < \psi_1(x) \quad (a < x < b).$$

Hence $G(x) = F(x) + x$ satisfies the required conditions.

**Lemma 4.** Let $H_1$, $H_2$ be disjoint $G_\delta$ sets and denote

(1) $E = [\text{cl} \ H_1 \setminus \text{int} \ (\text{cl} \ H_1)] \cup [\text{cl} \ H_2 \setminus \text{int} \ (\text{cl} \ H_2)]$.

Then

(a) $E$ is a nowhere dense closed set;

(b) if $(a, b)$ is any interval contiguous to $E$ then

(b1) $(a, b) \cap H_i \neq \emptyset$ implies that $(a, b) \cap H_i$ is everywhere dense in $(a, b)$, $i=1, 2$;

(b2) $(a, b) \cap H_1 \neq \emptyset$ implies $(a, b) \cap H_2 = \emptyset$.

**Proof.** (a) is trivial. For (b1) we observe that $(a, b) \cap \text{cl} \ H_i$ is open. In fact, $x \in (a, b) \cap \text{cl} \ H_i$ implies $x \notin E$, hence $x \in \text{int} \ (\text{cl} \ H_i)$ by the definition of $E$. But $(a, b) \cap \text{cl} \ H_i$ is also closed (relatively) in $(a, b)$, thus by the connectedness of $(a, b)$ either $\text{cl} \ H_i \cap (a, b) = \emptyset$ or $\text{cl} \ H_i \cap (a, b) = (a, b)$ holds which proves (b1). Finally if $H_i \cap (a, b) \neq \emptyset$, $H_2 \cap (a, b) \neq \emptyset$, then by (b1) both sets are everywhere dense in $(a, b)$.

Since $H_1$, $H_2$ are $G_\delta$ sets, $(a, b) \cap H_1 \cap H_2$ is everywhere dense in $(a, b)$ as well, and this contradicts the assumption $H_1 \cap H_2 = \emptyset$.

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