Chain Properties of Rule Closures

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Keywords: Term rewriting systems; Knuth–Bendix completion procedure; Divergence

Abstract. This article introduces a generalisation of the crossed rule approach to the detection of Knuth–Bendix completion procedure divergence. It introduces closure chains, which are special rule closures constructed by means of particular substitution operations and operators, as a suitable formalism for progress in this direction. Supporting substitution algebra is developed first, followed by considerations concerning rule closures in general, concluding with an investigation of closure chain properties. Issues concerning the narrowing process are not discussed here.

1. Introduction

In [HeP86] the importance of forward closures and later also backward closures for the definition of crossed rules was discovered. During the attempts to generalise the crossed rule notion it became apparent that their construction is strictly dependent on special forward or backward rule closures, called forward/backward chains. These closure chains are derived from rule closures by substitution variable constraints, and are constructed by means of special substitution operations and iterative operators. It is this iterative construction of the closure chains that makes them most suitable for the description of the Knuth–Bendix procedure divergence. Therefore attention is focused on a sufficiently general definition with the potential to describe the largest class of possible cases by their structural entities, followed by a deep investigation of their properties. Rather than offering a plethora of different isolated types of objects on the same level (although in examples the differences are strictly
pointed out), a synthetic approach has been applied, yielding a unifying view with a family of definitions interconnected by a duality principle.

The article is divided into four sections. Section 2 describes the motivation for investigations of closure chain properties on the basis of two divergence examples. Section 3 introduces the basic notation in term rewriting systems theory and provides the rule closure definitions – the main object of the article. Section 4 presents a special substitution algebra. In its first part it contains the definitions of two substitution operations – sum and join – with appropriate propositions proving their necessary properties. Upon the sum operation three iterative substitution operators – exponent, W- and T-operator – have been built and their properties have been proved in the second part of Section 4. Section 5 investigates the properties of rule closures. Its first part focuses on the structure and construction of rule closures in general, presenting a closure set generating procedure, and proving the closure set finiteness undecidability. The second part is introduced by general considerations concerning the construction of closure chains, followed by their definition. Section 5 is concluded by characterisation theorems of closure sets when they contain a closure chain, and by exact constructions of closure chains, taking advantage of the iterative operators from Section 4. An index containing all newly defined symbols in the paper is appended.

2. Motivation

Applying the Knuth–Bendix procedure to complete a given term rewriting system may result in the canonical term rewriting system, if it halts, or in a diverging process trying to generate an infinite set of rewrite rules. The first result concerning structural properties of an infinite set of rules generated by the Knuth–Bendix completion procedure is due to Huet [Hue81]. Further attempts to study the reasons of Knuth–Bendix procedure divergence through investigation of the structural properties of the generated rules were undertaken in [HeP86]. The notion of crossed rules as a sufficient criterion for proving divergence of the Knuth–Bendix procedure emerged from those investigations. Recently, complementary research by Kirchner [Kir89] focused on the description of infinite sets of rules generated by diverging Knuth–Bendix procedure by finite means, developing the notion of meta-rules and related objects.

The study of divergence patterns emerged from the observation of natural rewrite systems with diverging performance. Trying to enlarge the definition of crossed rules from [HeP86] so that it would cover a more general class of rewrite systems whose structural properties imply the divergence process appealed to the necessity of more insight into the behaviour of certain rewrite rule sequences.

Example 2.1. A natural example of a diverging rewrite system is associativity and endomorphism, studied in another context in [BeL87].

\[(x*y)*z \rightarrow x*(y*z)\]
\[f(x)*f(y) \rightarrow f(x*+y)\]

If we try to complete this system, choosing the operator precedence \(* > f\) with the left-to-right status of \(*\), we get a diverging process.