A Proof of a Relationship Between the Generalized Variances for Associated Autoregressive and Moving Average Processes

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Summary: In this paper we give a simple proof of the result that, for any integer $r$, given two processes of order $r$, one autoregressive and the other moving average but both with the same parameters, then the generalized variance of all orders $k \geq 2r$, for the autoregressive process, is exactly equal to the infinite order generalized variance for the moving average process.

1. Introduction

Define the $r$-th order autoregressive process and the associated $r$-th order moving average process by, respectively,

$$\sum_{j=0}^{r} \phi_j Z_{i-j} = A_i$$  \hspace{1cm} (1.1)

$$Z_i = \sum_{j=0}^{r} \phi_j A_{i-j}$$  \hspace{1cm} (1.2)

where $\{A_i\}$ denotes a sequence of uncorrelated identically distributed zero-mean random variables with unit variance$^2$), and the parameter set $(\phi_0, \phi_1, \ldots, \phi_r)$ is real with $\phi_0 = 1, \phi_r \neq 0$. However, we impose an extra restriction that the zeros of the polynomial $\sum_{j=0}^{r} \phi_j \xi^j$, in the complex variable $\xi$, must all lie outside the unit circle. This ensures that (1.1) is stationary, and that we can rewrite it in the form

$$Z_i = \sum_{j=0}^{\infty} \psi_j A_{i-j}$$  \hspace{1cm} (1.3)

where

$$\sum_{j=0}^{\infty} \psi_j \xi^j = \left( \sum_{j=0}^{r} \phi_j \xi^j \right)^{-1}$$  \hspace{1cm} (1.4)

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$^2$) The choice of unit variance is, as a referee remarks, rather unusual. But its use does avoid inelegantly cluttering up all the subsequent work with $\sigma_A^2$ and $\sigma_A^{-2}$. 
For any stationary process, including (1.1) and (1.2), the autocovariance at lag \( l \) is defined by

\[
\gamma_l = \text{Cov} [Z_{i+l} ; Z_{i+l}]
\]

and the \( k \times k \) matrix \( P_k = (p_{st}) \), where \( p_{st} = \gamma_{|s-t|} \), is termed the \( k \)-th autocovariance matrix. For processes (1.1) and (1.2), we will write \( P_k \) as, respectively, \( P_k (r, 0) \) and \( P_k (0, r) \). From (1.3), we see that, for the autoregressive process

\[
p_{st} = \sum_{j=0}^{\infty} \psi_j \psi_{j+|s-t|}
\]

which is convergent, due to the stationarity condition.

We will prove that, for all \( k \gg 2r \),

\[
\lim_{k\to\infty} |P_k (0, r) | = |P_k (r, 0) |
\]

a result suggested in Anderson [1976 a]. To do this, we shall need the following algorithm given in Anderson [1976b]:

If \( b = (b_{st}) \), when partitioned into submatrices \( b_{st} \), is a non-singular matrix with inverse \( c = (c_{st}) \), partitioned coherently for multiplication by \( b \), and if \( B \) differs from \( b \) in one square sub-matrix only, say the \( ST \)-th with \( \tilde{B}_{ST} = \tilde{b}_{ST} + \tilde{b}_{ST} \); then, provided

\[
\tilde{b}_{ST} \quad \text{and} \quad [\zeta_{TS} + \beta_{ST}^{-1}]
\]

are both regular, the inverse of \( B, = (C_{st}) \) say when partitioned in the same way as \( \zeta \), is given by the submatrices

\[
C_{st} = \zeta_{st} - \zeta_{sS} [\zeta_{TS} + \beta_{ST}^{-1}]^{-1} \zeta_{Tt}.
\]

This result is straightforwardly verified by showing that \((B_{st}) (C_{st}) = \mathbb{I}\), as we now demonstrate.

Define \( \beta_{st} = 0 \) whenever \( s \neq S \) or \( t \neq T \), then

\[
(B_{st}) (C_{st}) = (b_{st} + \beta_{st}) (\zeta_{st} - \zeta_{sS} [\zeta_{TS} + \beta_{ST}^{-1}]^{-1} \zeta_{Tt})
\]

\[= \mathbb{I} + (\sum b_{sj} \zeta_{jT} - \zeta_{jS} [\zeta_{TS} + \beta_{ST}^{-1}]^{-1} \zeta_{Tt}) \]

\[-(\sum b_{sj} \zeta_{jT} [\zeta_{TS} + \beta_{ST}^{-1}]^{-1} \zeta_{Tt})
\]

\[= \mathbb{I} + (\beta_{ST} (\zeta_{Tt} - \zeta_{TS} [\zeta_{TS} + \beta_{ST}^{-1}]^{-1} \zeta_{Tt}))
\]

\[-(\beta_{sS} [\zeta_{TS} + \beta_{ST}^{-1}]^{-1} \zeta_{Tt})
\]

where \( \beta_{sS} = \mathbb{I} \) if \( s = S \), but is null if \( s \neq S \). Thus