On Statistical Inference in Concentration Measurement

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Summary: The asymptotic distribution for a certain class of functionals of distribution functions is derived. This result is used to give distribution free asymptotic confidence intervals for these functionals; for this purpose, a strongly consistent estimate for the asymptotic variance is constructed. These results are applied to the Lorenz-curve and the Gini-measure as special cases of the above-mentioned class of functionals.

1. Introduction

Among the most frequently used quantities for the descriptive treatment of concentration phenomena are the Lorenz-curve (LC) and the Lorenz-concentration measure (LCM) (also called Gini-measure). If \( x^{(N)} = (x_1, \ldots, x_N) \) are the values of a positive attribute of a finite population of size \( N \), denote the LC by \( l(., x^{(N)}) \) and the LCM by \( \kappa (x^{(N)}) \); their definition is:

\[
l (\alpha, x^{(N)}) := \begin{cases} \\
\sum_{i=1}^{j} x_{Ni} / \sum_{i=1}^{N} x_i, & \text{for } \alpha = j/N, \ 1 \leq j \leq N \\
0, & \text{for } \alpha = 0
\end{cases}
\]

\[
l (\alpha, x^{(N)}) := l ((j - 1)/N, x^{(N)}) + N [l (j/N, x^{(N)}) - l ((j - 1)/N, x^{(N)})] (\alpha - (j - 1)/N) \quad \text{for } (j - 1)/N < \alpha < j/N, \ 1 \leq j \leq N
\]

\[
\kappa (x^{(N)}) = 1 - 2 \int_{0}^{1} l (\alpha, x^{(N)}) \, d\alpha = 1 - \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{i=1}^{k-1} x_{Ni} + x_{Nk} \right) / \sum_{i=1}^{N} x_i
\]

(1)

\[
\kappa (x^{(N)}) = \sum_{i=1}^{N} \left( 2 \frac{i}{N} - \frac{1}{N} - 1 \right) x_{Ni} / \sum_{i=1}^{N} x_i
\]

(2)

(\( x_{Ni}, 1 \leq i \leq N \), are the ordered components of \( x^{(N)} \)).

Note that the above definitions require the whole data \( x^{(N)} \); we will not deal with the corresponding definitions for grouped data during this paper.

There exists extensive literature on various concentration measures (for a broad presentation of these topics see Piesch [1975] and the references cited there), a considerable part of which is devoted to the discussion of their appropriateness for the description of concentration phenomena [cf. Bruckmann], their connection to related quantities like information measures, or their specific analytic properties [cf. Piesch]. It seems, however, that stochastic analysis of LC and LCM (to which we shall restrict our considerations) had been subject of investigations to a very poor amount only. This question becomes relevant when one is concerned with the description of concentration phenomena of an infinite population, e.g. on the basis of a random sample; then the attribute has to be considered as a random variable (r.v.) $X$ and "concentration" has clearly to be a property of the distribution function (d.f.) $F$ of $X$. Suppose $P \{X > 0\} = 1$ and $\mu_F := EX < \infty$, then the definitions corresponding to (1) and (2) in the present case are

$$l(\alpha, F) := \frac{\alpha}{\mu_F} \int_0^1 F^{-1}(t) \, dt, \quad 0 < \alpha \leq 1 \quad (1')$$

$$\kappa(F) := 1 - 2 \int l(\alpha, F) \, d\alpha = \frac{1}{\mu_F} \int_0^1 (2t - 1) F^{-1}(t) \, dt \quad (2')$$

(the last equality by partial integration), where, as usual, $F^{-1}(t) := \inf \{ \xi : F(\xi) \geq t \}$. Formally our problem will be as follows: let $X_i, 1 \leq i \leq n$, be i.i.d. r.v.-s distributed according to the d.f. $F$; based on this sample we intend to make statistical inference on functionals of d.f.-s like (1') (with fixed $\alpha$) or (2'). In particular we shall study the asymptotic properties of estimates for a slightly more general type of functionals (sec. 2) and will use these results to give distribution free asymptotic confidence bounds (sec. 3); the application of this procedure to the functionals (1') and (2') is treated in sec. 4.

2. An Asymptotic Normal Estimator

Let $(\Omega, A, P)$ be our basic probability space in the sequel, $X$ a r.v. with $P \{X > 0\} = 1$, $F$ its d.f., whence $F(0) = 0$ if we agree $F$ to be right continuous; denote by $F_*$ the set of d.f.-s with $F(0) = 0$, then, observing $\mu_F = \int_0^1 F^{-1}(t) \, dt$, the quantities (1') and (2') are functionals over $F_*$ of the type

$$\phi(F) := \frac{\int_0^1 J_c(t) F^{-1}(t) \, dt}{\int_0^1 J_d(t) F^{-1}(t) \, dt} \quad (3)$$