On the Property of Dullness of Pareto Distribution

By S. Talwalker, Bombay

1. Introduction

It has been observed, in the income distribution analysis, that the true income $X$ is usually under-reported. This phenomenon of under-reporting has been studied by Krishnaji [1970] and Revankar et al. [1974] among others. Krishnaji has assumed that the reporting errors are multiplicative, i.e. whenever an income is reported to be $T$, the true income $X$ is obtained as

$$T = RX \quad 0 < R \leq 1.$$ 

He has assumed this $R$ to be a random variable (rv), independent of $X$, having power function distribution and has obtained a characterization of Pareto type 1 distribution with pdf

$$f(x) = a x^{-a} \quad a > 0, \quad x \geq \xi > 0.$$  \hspace{1cm} (1.1)

Whenever $\xi = 1$, the pdf becomes

$$f(x) = ax^{-a} \quad a > 0, \quad x \geq 1.$$ \hspace{1cm} (1.2)

Revankar et al. [1974] assume that the under-reporting errors are additive, i.e. $T = X - V$ where $V$ is a rv such that $E(V/X = x) = \alpha + bx$ with $0 < b < 1$ and $a = -bm$ where $m$ denotes the tax exempt level.

They have shown that the condition

$$E(V/x > y) = \alpha + \beta y$$

is necessary and sufficient for the rv $X$ to follow Pareto distribution.

In this note, it is assumed that the incomes are under-reported, and that the reporting errors are multiplicative. We assume that all reported incomes are $\geq m$ where $m$ denotes the tax-exempt level, since there is no 'incentive' to report, whenever the income falls below $m$, see Revankar et al. [1974].

Under these assumptions 2 characterizations of Pareto type 1 distribution given by (1.2) are obtained. They are presented in section 2.

\hspace{1cm} 1) Dr. Sheela Talwalker, CIBA-GEIGY Research Centre, Bombay 400 063, India.
2. Main Theorem

Let the rv $X$ represent the true incomes in a society, such that these true incomes are usually under-reported. Further let these reporting errors be multiplicative of the form

$$X = TS$$

where the rv $S$ denotes the under-reporting error, $T$ denotes the reported income and $X$ denotes the true income. It follows therefore that

$$S \in [1, \infty).$$

Now, let us assume that $X$ is a rv with a continuous cumulative distribution function (cdf) $F(x)$, such that $F(x) = 0$ for $x < 1$. Consider the conditional probability that the true income $X \geq x$, where $x$ is some real number $\geq t$, where $t$ denotes the reported income, in a given particular situation. It is obtained as

$$\Pr (X \geq x/X \geq t).$$

Since $x \geq t$, writing $x = st$ where $s \geq 1$, the conditional probability can be written as

$$\Pr (X \geq st/X \geq t) = \frac{G(st)}{G(t)} \quad s \geq 1$$

(2.1)

$$G(s) = 1 - F(s) \quad s \geq 1.$$

Definition: An income distribution of $X$ is called dull at a point $t$, i.e. incapable of utilizing the information about the reported income $t$, whenever

$$\Pr (X \geq st/X \geq t) = \Pr (X \geq s) \quad \text{for all } s \geq 1 \text{ and given } t$$

(2.2)

i.e. $\frac{G(st)}{G(t)} = G(s)$.

The distribution of $X$ will be called totally dull, if equation (2.2) holds true for all $s \geq 1$ and all $t \geq m > 1$ where $m$ is some fixed real number, denoting the tax exempt level.

The property of dullness of an income distribution can be interpreted as follows: the conditional probability that the true income $X$ is at least $s$ times the reported value $t$ ($s \geq 1$), is the same as the unconditional probability that $X$ is at least $s$. This holds true for all $s \geq 1$.

In other words, the distribution of error in an income, reported to be $t$, (i.e. $X/t = S$), is the same as that of $X$, or the reporting error in an income is independent of its reported value.

This property of dullness of Pareto distribution, as an income distribution, is parallel to the famous property of 'forgetfulness' or 'lack of memory' of the exponential distribution, as a life distribution.