Bivariate Non-central Negative Binomial Distribution: Another Generalisation

By S. H. Ong and P. A. Lee

Summary: Another bivariate generalisation (Type V) of the non-central negative binomial distribution is considered. This generalisation is constructed (i) as a latent structure model; (ii) as an extension of an accident proneness model investigated by Edwards/Gurland (1961); and (iii) as a reversible stochastic counter model. The third construction gives, as a result, an apparently new formulation of the Edwards/Gurland model. The probabilities, moments, recurrence formulas and some properties are given. An application to the data used by Holgate (1966) is considered.

1 Introduction

Bivariate generalisations of the non-central negative binomial (NNB) distribution (Types I, II, III and IV) were introduced by Lee/Ong. In this paper we show that these generalisations (except the Type III) can be obtained systematically from a latent structure scheme (Goodman) and thereby obtain another bivariate generalisation of the NNB. We designate this as the Type V distribution.

The Type V distribution is also formulated (i) as an extension of an accident proneness model investigated by Edwards/Gurland, and (ii) as a reversible stochastic counter model studied by Lampard (1968) and Lee. Edwards/Gurland called their model the compound correlated bivariate Poisson (CCBP) distribution. The CCBP distribution was also studied by Subrahmaniam, Mitchell/Paulson with additional properties mentioned by Ong/Lee (1982). In deriving the Type V distribution as a reversible stochastic counter model we have, as a result, obtained a stochastic formulation for the CCBP distribution; this is believed to be new.

The probabilities, moments and recurrence formulas are obtained together with discussions on some properties. In particular the distribution of sum is found to extend the Gegenbauer distribution of Plunkett/Jain, and the conditional distribution is seen to be related to a family of distributions considered by Kemp (1968). As an application the data used by Holgate (1966) is considered using moments estimation. For the purpose of comparison the other bivariate generalisations are also fitted to the data; the fits seem to be adequate.

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2 Formulations of the Bivariate NNB Distributions

2.1 Latent Structure Models

Consider two random variables $X$ and $Y$ each with the common parameter $\theta$ where $\theta$ is a random variable $\Theta$. A model derived in this manner is termed a "latent structure model" (Goodman). $X$ and $Y$ can be regarded as alternative measures of the same thing. As a generalisation let $X$ and $Y$ have the parameters $\theta_1$ and $\theta_2$ respectively with $\Theta_1$ and $\Theta_2$ jointly distributed.

Using the above scheme the Type I and Type IV distributions are constructed as follows:

**Type I**

Let $X$ and $Y$ be independently distributed negative binomial variables with parameters $(\nu_1 + r, p_1)$ and $(\nu_2 + r, p_2)$ respectively where $r$ is a Poisson variable $R$ with probability distribution (pd)

$$P(R = r|\lambda) = e^{-\lambda} \frac{\lambda^r}{r!}, \quad \lambda > 0.$$ 

The unconditional probability generating function (pgf) (see Lee/Ong) is

$$G(u, v) = \left( \frac{q_1}{1 - p_1 u} \right)^{\nu_1} \left( \frac{q_2}{1 - p_2 v} \right)^{\nu_2} \exp \left\{ \lambda \left[ \left( \frac{q_1}{1 - p_1 u} \right) \left( \frac{q_2}{1 - p_2 v} \right) - 1 \right] \right\}.$$  \hspace{1cm} (2.1)

**Type IV**

Now let the parameters of $X$ and $Y$ be $(\nu_1 + r_1, p_1)$ and $(\nu_2 + r_2, p_2)$ respectively with $(r_1, r_2)$ jointly distributed as the bivariate Poisson $(R_1, R_2)$ with joint pd

$$P(R_1 = r_1, R_2 = r_2) = \sum_{n=0}^{\min(r_1, r_2)} \frac{(\lambda_1 - \rho)^{r_1-n} (\lambda_2 - \rho)^{r_2-n} \rho^n e^{-(\lambda_1 + \lambda_2 - \rho)}}{(r_1-n)! (r_2-n)! n!}$$ \hspace{1cm} (2.2)