PERFECTLY PLASTIC STRESS FIELD AT
A STATIONARY CRACK TIP*

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Abstract

Under the hypothesis that all the perfectly plastic stress components at a crack tip are the functions of \( \theta \) only, making use of yield conditions and equilibrium equations, we derive the generally analytical expressions of the perfectly plastic stress field at a crack tip. Applying these generally analytical expressions to the concrete cracks, the analytical expressions of perfectly plastic stress fields at the tips of Mode I, Mode II, Mode III and Mixed Mode I - II cracks are obtained.

I. Introduction

A problem on the perfectly plastic stress field at a stationary crack tip was studied by Hutchinson\[1\], Shih\[2\], as well as Hult and McClintock\[3\]. However, they did not give an overall and full discussion about this problem. In addition, the perfectly plastic stress fields at mode II and mixed mode I - II plane-stress crack tips all are not given in references. For this reason, in this paper, we propose a very simple method to solve the above-mentioned problems.

Under the hypothesis that all the perfectly plastic stress components at a crack tip are the functions of \( \theta \) only, making use of yield conditions and equilibrium equations, we derive the generally analytical expressions of perfectly plastic stress field at a crack tip. Let us apply these generally analytical expressions to the concrete cracks. The analytical expressions of perfectly plastic stress field at the tips of Mode I, Mode II, Mode III and Mixed Mode I - II cracks are obtained. Some results in this paper are identical with those in [1], [2] and [3], it is shown that the method proposed by the author is correct.

II. Anti-plane Shear

We locate the origin of a polar coordinate system \((r, \theta)\) in a crack tip \(O\), as shown in Fig. 1.

For the case of anti-plane shear, assuming that all the shearing stress components \(\tau_r\) and \(\tau_\theta\) are the functions of \( \theta \) only, then equilibrium equation becomes

\[
\frac{d\tau_\theta}{d\theta} + \tau_r = 0
\]  

(2.1)


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\[ \tau_r^2 + \tau_\theta^2 = k^2 \]  

in which \( k \) is the shear yield limit.

By using (2.1) and (2.2), we derive the generally analytical expressions of perfectly plastic stress field at Mode III crack tip, as

\[ \tau_\theta = \pm k, \quad \tau_r = 0 \]  

and

\[ \tau_r = a_1 \sin \theta + a_2 \cos \theta, \quad \tau_\theta = a_1 \cos \theta - a_2 \sin \theta \]  

where \( a_1 \) and \( a_2 \) are two arbitrary constants.

Applying (2.3) and (2.4) to Mode III crack directly, we obtain the analytical expressions of perfectly plastic stress field at Mode III crack tip, as

\( \hat{r} \) : \( 0 \leq \theta \leq \frac{\pi}{2} \) \quad \tau_r = 0, \quad \tau_\theta = k \]  

(2.5)

\( \pi/2 \leq \theta \leq \pi \) \quad \tau_r = -k \cos \theta, \quad \tau_\theta = k \sin \theta \]  

(2.6)

This result is identical with that in [3].

III. Plane Strain

Assuming that all the stress components \( \sigma_r, \sigma_\theta, \tau_{r\theta} \) at a crack tip are the functions of \( \theta \) only, then equilibrium equations become

\[ \frac{d\tau_{r\theta}}{d\theta} + \sigma_r - \sigma_\theta = 0, \quad \frac{d\sigma_\theta}{d\theta} + 2\tau_{r\theta} = 0 \]  

and the yield condition under plane strain is

\[ (\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2 = 4k^2 \]  

(3.1)

(3.2)

where \( \sigma_r, \sigma_\theta \) are the normal components, and \( \tau_{r\theta} \) the tangential component of stress.

Making use of (3.1) and (3.2), we obtain the generally analysis expressions of perfectly plastic stress field at a plane-strain crack tip, as

\[ \tau_{r\theta} = \pm k, \quad \sigma_r = \sigma_\theta = b_1 \mp 2k(\theta - \theta_0) \]  

(3.3)

and

\[ \begin{align*}
\sigma_r &= -b_2 \cos 2(\theta - \theta_0) + b_3 \sin 2(\theta - \theta_0) + b_4 \\
\tau_{r\theta} &= b_2 \sin 2(\theta - \theta_0) + b_3 \cos 2(\theta - \theta_0) + b_4
\end{align*} \]  

(3.4)

where \( b_i \) (\( i = 1 - 4 \)) are four — integral constants, and \( \theta_0 \) is an indeterminate constant.

If there exists a radial line of stress discontinuity for the perfectly stress field at a plane-strain crack tip, we will have\([3]\):

\[ \sigma_r^* = \sigma_\theta^*, \quad \tau_{r\theta}^* = \tau_{r\theta}^*, \quad \sigma_r^* = \sigma_\theta^* = 4\sqrt{k^2 - \tau_{r\theta}^2} \]