HOMOGENIZED EQUATIONS FOR STEADY HEAT CONDUCTION IN COMPOSITE MATERIALS WITH DILUTELY-DISTRIBUTED IMPURITIES*

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ABSTRACT

In this paper, by using the two-space method, homogenized equations for steady heat conduction in the composite material cylinders with dilutely-distributed elliptic cylinders of impurities are derived, and the explicit expressions for the corresponding effective heat conductivity of those which are concerned are obtained. It is also shown that the macroscopic heat conduction is anisotropic when the cross-sections of the impurity cylinders are unidirectionally oriented and isotropic when the angular distribution of the cross-sections is uniform.

I. Introduction

As is well-known, the physical properties of composite materials frequently exhibit small-scale variations, and hence, it is difficult to describe and analyze them. It is of practical engineering interest to give an effective macroscopic description of the properties with the help of appropriate homogenization procedure. J.B. Keller [1], [2] made homogenization treatments of heterogeneous media by using the two-space method and the smoothing method. Employing the former method, L. Ting [3] studied the heat conduction problems in the media with small spherical impurities. In the present paper, following Keller's procedure, we shall derive the homogenized equations for steady heat conduction in a kind of composite material cylinders with dilutely-distributed impurities and obtain the explicit expressions for the effective heat conductivities. Let us consider the problem with the help of the following model, assuming that

1. an infinitely long cylinder is embedded in dilutely-distributed elliptic cylinders of impurities, whose total number and (surface) number density are $N$ and $n$, respectively.

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(2) both the semi-axes of the cross-sections of the impurity cylinders are
\( a \) and \( b \), where \( a \geq b \), \( a \) and \( b \) are of the same order as the transverse dimension of the composite material cylinder, whereas \( e \) is a small parameter, i.e., \( 0 < e \ll 1 \).

(3) the heat conductivity of all the impurity cylinders is constant \( k_a \),
while that outside them is constant \( k_b \) (\( k_a \neq k_b \)).

(4) there is a steady heat source with the strength of \( h(x, \frac{x}{e}, e) \) throughout
the cylinder, inside which the steady temperature distribution is \( u(x, e) \), where \( x = (x_1, x_2) \) are the transverse coordinates of the points within the cylinder.

In the following, we shall first derive the general homogenized equations for the case considered. Then we shall deal with two special cases. One of them illustrates the unidirectional orientation of the cross-sections of the impurity cylinders, and the other illustrates the uniform angular distribution of the cross-sections. We conclude that the macroscopic heat conduction is anisotropic for the former, and isotropic for the latter.

II. Homogenization Procedure

In this section, we shall follow Keller's homogenization procedure\(^[2]\) to derive the homogenized equation for steady heat conduction and effective heat conductivities.

In the case treated, the governing equation is
\[
\frac{\partial}{\partial x_1} \left( K \frac{\partial u}{\partial x_1} \right) = h \left( x, \frac{x}{e}, e \right), \quad x = (x_1, x_2) \in D
\]
where
\[
K = \begin{cases} 
  k_1 & x \in D, \quad (j = 1, 2, \ldots, N) \\
  k_0 & x \in D, \quad x \in D_i
\end{cases}
\]
\( D \) is the domain of the cross-section of the whole cylinder, \( D_i \) \( (j = 1, 2, \ldots, N) \) are those of the impurity cylinders (cf. Fig.1), and the summation convention has been adopted here and below.

Introduce the two-space scales
\[
x = \frac{x}{e}
\]
and set
\[
u(x, y, e) = u(x, y, e) = v_0(x, y) + \varepsilon v_1(x, y) + \varepsilon^2 v_2(x, y) + \ldots
\]
\[
h(x, \frac{x}{e}, e) = h_0(x, y) + \varepsilon h_1(x, y) + \varepsilon^2 h_2(x, y) + \ldots
\]
Substituting them into eq. (2.1) and equating the coefficients to the same powers of \( e \), we obtain the recursive equations
\[
L_1 v_0 = 0
\]