FIXED POINT THEOREMS IN PROBABILISTIC INNER PRODUCT SPACES*

Zhang Shi-sheng

(Department of Mathematics, Sichuan University, Chengdu).

(Recieved Mar. 8, 1984)

Abstract

The purpose of this paper is to introduce a new modified definition for probabilistic inner product space, and to establish several new fixed point theorems for mappings on such kind of spaces. As an example of applications, we utilize the results of this paper to study the existence and uniqueness of solution of Uryson's integral equation in IV.

It is well known that theory and applications of probabilistic metric spaces are of fundamental importance in the theory of probabilistic functional analysis. Recently, the concept of probabilistic inner product space has been introduced and considered by Schweizer and Sklar[1], as well as Dumitrescu[2]. Owing to the complication of the original definition concerning the probabilistic inner product space, the theory of such kind of spaces is too difficult to develop and investigate.

The purpose of this paper is to introduce a new modified definition for probabilistic inner product space, and establish several new fixed point theorems for mappings in this kind of spaces. As an example of applications, we utilize the results of this paper to study the existence and uniqueness of the solution of Uryson's integral equation.

I. Definitions and Examples of Probabilistic Inner Product Spaces

Throughout this paper let \( R = ( -\infty, +\infty ) \), \( R^* = [0, \infty) \).

**Definition 1** A mapping \( F: R \rightarrow R^* \) is called a distribution function, if it is nondecreasing, left-continuous with

\[
\inf_{t \in R} F(t) = 0, \quad \sup_{t \in R} F(t) = 1.
\]

In what follows we always denote by \( \mathcal{D} \) the set of all distribution functions, and denote by \( H \) the function defined by

\[
H(t) = \begin{cases} 
1 & (t > 0) \\
0 & (t \leq 0)
\end{cases}
\]

**Definition 2** A mapping \( \mathcal{A}: [0,1] \times [0,1] \rightarrow [0,1] \) is called a strong t-norm, if it satisfies the following conditions:

* Communicated by Zhou Mo-ren.
Example 1. We list here two of the simplest t-norms:

- Product, i.e., \( t_1(a, b) = a \cdot b, \) \( \forall a, b \in [0, 1] \)
- Min, i.e., \( t_2(a, b) = \min\{a, b\}, \) \( \forall a, b \in [0, 1] \)

In the sequel, we shall denote by \( \tilde{J} \) the dual norm of the strong t-norm \( J \), i.e.
\[
\tilde{J}(a, b) = J(1 - a, 1 - b), \quad \forall a, b \in [0, 1].
\]

**Definition 3** A probabilistic inner product space \((E, \mathcal{F}, \langle \cdot, \cdot \rangle)\) is an ordered triplet, where \( E \) is a real linear space, \( \mathcal{F} \) is a strong t-norm, \( \langle \cdot, \cdot \rangle \) is a mapping of \( E \times E \rightarrow [0, 1] \) satisfying the following conditions (we shall denote the distribution function \( F(x, y) = \mathbb{P}(X \leq x, Y \leq y) \) will represent the value of \( F_{x, y} \) at \( t \in \mathbb{R} \))

\[
\begin{align*}
(PI-1) & \quad F_{x, y} = F_{y, x}, \quad \forall x, y \in E; \\
(PI-2) & \quad F_{x, y}(0) = 0, \forall x \in E; \text{ and } F_{x, y}(t) = t(t), \forall t \in \mathbb{R}, \text{ if and only if } x = y; \\
(PI-3) & \quad F_{x, y}(t_1) \geq F_{x, y}(t_2) \geq t(1 - t_1) + t(1 - t_2) + t(x_1 + y_2) \geq F_{x, y}(t_1) + F_{x, y}(t_2), \\
& \quad \forall x_1, y_1, x_2, y_2 \in E, \quad t_1, t_2 \in \mathbb{R}; \\
(PI-4) & \quad \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = 1, \quad \forall x, y \in E; \\
(PI-5) & \quad \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} + \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\}.
\end{align*}
\]

Example 2. Let \((E, (\ldots))\) be a common inner product space. Then the space \((E, \mathcal{F}, \langle \cdot, \cdot \rangle)\) induced by \((E, (\ldots))\) is a special probabilistic inner product space, where \( J_2 = \min \), and \( \mathcal{F} \) is a mapping of \( \times \times E \rightarrow [0, 1] \), which is defined by
\[
F_{x, y}(t) = H(t - (x, y)), \quad \forall x, y \in E, \quad t \in \mathbb{R}
\]

**Proposition 1** Let \((E, \mathcal{F}, \langle \cdot, \cdot \rangle)\) be a probabilistic inner product space. Then the function \( \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} \) has the following properties:

\[
\begin{align*}
( i ) & \quad \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\}, \quad \forall x, y \in E; \\
( ii ) & \quad \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} \geq 0; \text{ and } \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = 0 \\
& \quad \text{if and only if } x = y; \\
( iii ) & \quad \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\}, \quad \forall a \in \mathbb{R}; \\
( iv ) & \quad \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} + \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\}, \quad \forall x, y \in E.
\end{align*}
\]

**Proof.** (i) and (iii) follow from (PI-1) and (PI-5) immediately. Besides, by condition (PI-2), \( F_{x, y}(0) = 0, \) \( \forall x \in E \), hence we have \( \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = 0, \forall x \in E \). Furthermore, if \( \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = \emptyset \), then for any \( \varepsilon > 0 \), \( F_{x, y}(\varepsilon) = 1 \). By the nondecreasing property of distribution functions, and \( F_{x, y}(0) = 0 \), \( \forall x \in E \), it gets \( F_{x, y}(t) = H(t), \) \( \forall t \in \mathbb{R} \), so that from (PI-2) we have \( x = y \).

Conversely, if \( x = y \), it follows from (PI-2) that
\[
\sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} = \sup\{t \in \mathbb{R}, H(t) < 1\} = 0
\]

This proves (ii).

Letting \( a(x, y) = \sup\{t \in \mathbb{R}, F_{x, y}(t) < 1\} \), and using (PI-3) for any \( \varepsilon > 0 \) we have