A UNIFORMLY CONVERGENT DIFFERENCE SCHEME FOR
THE SINGULAR PERTURBATION OF A SELF ADJOINT
ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

Liu Fa-wang (刘发旺)
(Fuzhou University, Fuzhou)
Zheng Xiao-su (郑小苏)
(Fujian Teachers' University, Fuzhou)

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Abstract

In this paper, we consider a self adjoint elliptic first boundary value problem with a small parameter affecting the highest derivative.

In the paper, we set up a new scheme by the asymptotic analysis method, compare asymptotic behavior between the solution of the difference equation and the solution of the differential equation, and show uniform convergence of the new scheme.

I. Introduction

In this paper, we discuss a uniformly convergent difference method for solving the first boundary value problem of the self adjoint elliptic partial differential equation with a small parameter affecting the highest derivative:

\[ \begin{aligned}
L_\epsilon u_\epsilon (x, y) &= -\epsilon \frac{\partial^2 u_\epsilon}{\partial x^2} - \epsilon \frac{\partial^2 u_\epsilon}{\partial y^2} + b(x, y) u_\epsilon = f(x, y) \\
\quad u_\epsilon (x, y) \big|_{\partial \Omega} &= 0
\end{aligned} \]  

in the region of square \( \bar{\Omega} \) \( (0 \leq x \leq L; \ 0 \leq y \leq T) \), where \( \epsilon > 0 \) is a small parameter, \( b(x, y) \geq \beta > 0 \).

When \( \epsilon = 0 \), eqs. (1.1), (1.2) are degenerated into the problem:

\[ \begin{aligned}
L_0 W(x, y) &= b(x, y) W(x, y) = f(x, y) \\
W(x, y) &= f(x, y) / b(x, y)
\end{aligned} \]  

(1.3)

For such a problem, we hope to construct a difference scheme that can adapt to the behavior of boundary layer, that is to say, for the fixed mesh size, when \( \epsilon \to 0 \), the asymptotic behavior of the solution of the difference equation is consistent with that of differential equation. What interests us most is that when \( h \to 0 \), \( \tau \to 0 \), the solution of the scheme constructed can converges uniformly in \( \epsilon \) to the solution of differential equation. In the last ten years, many people have done this kind of work, for example, Doolan, Miller, Schilders et al, but their discussions are largely on the ordinary differential equation with a small parameter.
In this paper, according to the behavior of boundary layer of the differential equation we construct the corresponding difference scheme, analyse the asymptotic behavior of the solution, and prove the uniform convergence of the solution.

II. Setting up the Difference Equation

By the analysis of asymptotic method, we can see that, when $\varepsilon = 0$, all definite conditions of the differential problems (1.1), (1.2) are lost at the four sides $x = 0, x = L, y = 0, y = T$. At this moment, when $\varepsilon \to 0$, solution $u_{e}(x, y)$ of perturbation problems (1.1), (1.2) cannot uniformly approximate solution $W(x, y)$ of reduced problem (1.3) in the layers. In other words, there will be a phenomenon of boundary layers in the neighborhood of sides $x = 0, x = L, y = 0, y = T$. Thus in problems (1.1), (1.2) exist the boundary layers functions on $\partial \Omega$:

\begin{align*}
V^{(1)} &= \exp \left( -\sqrt{\frac{b}{\varepsilon}} \frac{x}{\sqrt{\varepsilon}} \right), \\
V^{(2)} &= \exp \left( -\sqrt{\frac{b}{\varepsilon}} \frac{L-x}{\sqrt{\varepsilon}} \right), \\
V^{(3)} &= \exp \left( -\sqrt{\frac{b}{\varepsilon}} \frac{y}{\sqrt{\varepsilon}} \right), \\
V^{(4)} &= \exp \left( -\sqrt{\frac{b}{\varepsilon}} \frac{T-y}{\sqrt{\varepsilon}} \right)
\end{align*}

All of these are exponential functions.

Thus, we can construct a difference scheme:

\begin{equation}
\frac{\partial^{2} u_{e}}{\partial x^{2}} + \frac{\partial^{2} u_{e}}{\partial y^{2}} + b(x, y) u_{e}(x, y) = f(x, y) \tag{2.2}
\end{equation}

Where $u_{e}^{(h, r)}$, $u_{e}^{(h, r)}$ are second-order central difference quotients relative to $x$, $y$, respectively, $h$, $r$ are step sizes in $x$, $y$, direction respectively, $nh = L$, $nr = T$, and $\gamma_{1}$, $\gamma_{2}$ are indeterminate coefficients.

In order to determine $\gamma_{1}$, $\gamma_{2}$, when $b$ is a constant, we require that $V^{(i)}$ $(i = 1, 2, 3, 4)$ satisfy constant coefficient homogeneous difference equation:

\begin{equation}
-\gamma_{1} u_{e}^{(h, r)} - \gamma_{2} u_{e}^{(h, r)} + b u_{e}^{(h, r)} = 0 \tag{2.3}
\end{equation}

Let

\begin{equation*}
\begin{aligned}
u^{(h, r)}(x, y) &= \exp \left( -\sqrt{\frac{b}{\varepsilon}} \frac{x}{\sqrt{\varepsilon}} \right), \text{ or } \\
u^{(h, r)}(x, y) &= \exp \left( -\sqrt{\frac{b}{\varepsilon}} \frac{L-x}{\sqrt{\varepsilon}} \right)
\end{aligned}
\end{equation*}

substituting into (2.3), we obtain

\begin{equation*}
\gamma_{1} = \frac{1}{4} b h^{2} \sinh \left( \frac{\sqrt{\frac{b}{\varepsilon}} h}{2} \right)
\end{equation*}

In the same way, let

\begin{equation*}
\begin{aligned}
u^{(h, r)}(x, y) &= \exp \left( -\sqrt{\frac{b}{\varepsilon}} \frac{y}{\sqrt{\varepsilon}} \right), \text{ or } \\
u^{(h, r)}(x, y) &= \exp \left( -\sqrt{\frac{b}{\varepsilon}} \frac{T-y}{\sqrt{\varepsilon}} \right)
\end{aligned}
\end{equation*}

substituting into (2.3), we have

\begin{equation*}
\gamma_{2} = \frac{1}{4} r^{2} \sinh \left( \frac{\sqrt{\frac{b}{\varepsilon}} r}{2} \right)
\end{equation*}