A NEW QUADRILATERAL NONCONFORMING MODEL
AND ITS CONVERGENCE

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Abstract

This paper presents a new quadrilateral nonconforming finite element. We deal with its
convergence by using a generalized patch test. The error on stresses and displacements are
obtained and numerical computations for plane elast problems are given.

I. Introduction

Q4 element is the simplest among quadrilateral isoparametric elements. It fits arbitrary meshes,
and has extensive application in engineering. However, its numerical solutions are not ideal.
Especially, such solutions don't reflect plane deformations very well.

By increasing inner degrees of freedom of quadrilateral isoparametric element Q4, Wilson et al
constructed a nonconforming displacement model Q6, which violated compatibility of elements
and brought about satisfactory results. Generally, however, Q6 fits rectangular and parallelograms
meshes only. Convergence is not guarantees for arbitrary meshes and therefore its application is
limited.

Many scholars have been working on the problems of the convergence of nonconforming
elements. According to the experience of practical calculation and the background of mechanics,
Inongs put forward the patch test, which promoted the development of nonconforming finite
elements. Describing the patch test mathematically, Strang and Fix believe that it is the necessary and
sufficient condition for convergence of nonconforming elements. But, theory and practical
computations indicated that patch test is neither necessary nor sufficient for convergence of
nonconforming elements. To find a necessary and sufficient condition for convergence of
nonconforming finite elements, Stummel presented so-called generalized patch test and proved its
necessity and sufficiency.

By means of energy test criterion (a stronger form of Inongs's patch test), [11] constructed a
quadrilateral nonconforming model QMS. Using Stummel's generalized patch test, we proved
theoretically that QMS is convergent for arbitrary quadrilateral meshes, and obtained the following
error estimates for stresses and displacements:

$$\|u - u_h\|_h = O(h)\quad \|u - u_h\|_0 = O(h^2)$$

Finally, a few examples of numerical computations are presented. Results of computations
indicate that QMS not only improves numerical properties of Q4, but also fits irregular meshes.

* [10] proved that Q6 is convergent when certain restrictions are inserted on meshes.
II. Basic Notions and Lemmas

Let $G$ be a polygonal domain in $\mathbb{R}^2$, and $K_h$ be a subdivision of $G$ by convex quadrilaterals $K$ satisfying the regularity condition ([2], p.247).

Let $K$ be a convex quadrilateral having the vertices $P_i = (x_i^1, x_i^2) \ (1 \leq i \leq 4)$, and let $\hat{K} = [-1, 1] \times [-1, 1]$ be the reference square having the vertices $\hat{P}_i = (1 \leq i \leq 4)$ (Fig. 1). Then there exists a unique mapping $F_K \in (Q_1(\hat{K}))^2$ given by

$$x_j = \sum_{i=1}^{4} x_i^j N_i(\zeta, \eta) \quad (j = 1, 2) \quad (2.1)$$

Such that

$$F_K(P_i) = P_i \quad (1 \leq i \leq 4), \quad F_K(K) = K$$

where

$$N_i(\zeta, \eta) = \frac{1}{4} (1 + \zeta_i \hat{\zeta})(1 + \eta_i \hat{\eta}) \quad (1 \leq i \leq 4)$$

$Q_1(\hat{K})$ is the space of bilinear polynomials on $\hat{K}$.

![Fig. 1](image)

It is known that under the regularity assumption for subdivision $K_h$, there exist constants $c_1$, $c_2$ such that

$$c_1 |v|_1, K \leq |\hat{v}|_1, K \leq c_2 |v|_1, K \quad (2.2)$$

$$|\hat{v}|_2, K \leq c h_K |v|_2, K \quad (2.3)$$

for every $\hat{v} \in H^2(\hat{K})$ and the associated $v = \hat{v} \cdot F_K^{-1} \in \tilde{H}^2(K)$ (See [2]).

Quadrilateral nonconforming element $Q_{K_h}$ is defined as follows: On the reference square $\hat{K}$, the shape function is of the form

$$\hat{a}(\zeta, \eta) = \sum_{i=1}^{4} \hat{u}_i N_i(\zeta, \eta) + \{10[(\zeta^4 - 1) + (\eta^4 - 1)] - 9[(\zeta^2 - 1) + (\eta^2 - 1)]\} t \quad (2.4)$$

which gives the shape function on quadrilaterals $K \in K_h$ by

$$u = \hat{a} \cdot F_K^{-1} \quad (2.5)$$

therefore, the shape function on $K$ is uniquely determined by its values $u_i$ at the vertices $P_i \ (1 \leq i \leq 4)$ of $K$ inner parameter $t$.