STRUCTURE AND RIGIDITY
IN HYPERBOLIC GROUPS I

E. RIPS AND Z. SELA

Abstract

We introduce certain classes of hyperbolic groups according to their possible actions on real trees. Using these classes and results from the theory of (small) group actions on real trees, we study the structure of hyperbolic groups and their automorphism group.

In [Gr1] M. Gromov introduced hyperbolic groups and showed how geometric notions, tools, and results, mostly from the theory of negatively curved manifolds, can be adapted to obtain deep and broad algebraic results on the structure of hyperbolic groups and their subgroups. Gromov’s paper and a recent work of the second author on the isomorphism problem [Se1] stress the need for understanding the structure of automorphisms of hyperbolic groups and more globally the structure of the automorphism group of a hyperbolic group.

In this paper we adapt results from the work of the first author on group actions on real trees [R] to the study of hyperbolic groups and their automorphisms. Our approach is an elaboration of the Bestvina-Paulin method ([B], [P]) and we believe that besides the results we obtain our arguments should be applicable for future problems. The results we get serve as key points in our solution to the isomorphism problem [Se2].

We start by introducing certain classes of hyperbolic groups in terms of their possible actions on real trees. This classification, although very simple, turns out to be essential in understanding automorphisms and may serve as possible induction steps for future problems. In sections 2 and 3, we bring an immediate application of the Bestvina-Paulin method for the Hopf and co-Hopf properties for certain hyperbolic groups.

The automorphism group of a surface group is generated by Dehn twists and inner automorphisms. The notion of a Dehn twist can be generalized

---

The second author was partially supported by an NSF grant.
to an arbitrary group as an automorphism inherited from a splitting of the group into an amalgamated product or a $HNN$ extension:

(i) if $\Gamma = A \ast_C B$ and $c$ is central in $C$, we set:

$$\forall a \in A \quad \varphi(a) = a$$

$$\forall b \in B \quad \varphi(b) = cb \, c^{-1}$$

(ii) if $\Gamma = A \ast_C = \langle A, t | t \alpha(c) t^{-1} = \beta(c) \rangle$ and $c$ is central in $C$, we set:

$$\forall a \in A \quad \varphi(a) = a$$

$$\varphi(t) = t(\alpha(c))$$

We call an automorphism obtained by a sequence of Dehn twists and inner automorphisms internal (the notion was suggested to us by Benjamin Weiss). In section 4 we start developing our machinery in order to show that for torsion-free hyperbolic groups, the group of internal automorphisms is of finite index. We do this by constructing a real tree equipped with isometric group action in case the index of the internal subgroup is infinite and then show in sections 5 and 6 such a real tree cannot be obtained by our construction. Having a complete "proof scheme", we show how to get Gromov's theorem on freely indecomposable subgroups in section 7, and in section 8 we prove the automorphism group of a hyperbolic group is finitely generated. Section 9 brings a classification of what we call "weakly rigid" groups in terms of the existence of "quadratically hanging" subgroups. The importance of these notions becomes much clearer in [Se3]. At the end of this paper we bring a short appendix which summarizes some of the main results in the theory of (small) group actions on real trees. We refer to that appendix throughout the paper. A reader who is not yet familiar with this theory, may prefer to start with the appendix.

Further structural results on hyperbolic groups, their small splittings and automorphism group appear in a continuation paper by the second author [Se3]. Application of the techniques presented in this paper to (acylindrical) accessibility of finitely generated and finitely presented groups appears in [Se4].

1. Rigidity Properties

Actions of groups on real trees suggest a natural hierarchy of classes of groups which turns out to be essential in studying the structure of hyperbolic groups and their automorphism groups. This hierarchy seems to be a key