HEAT KERNEL AND HARDY ESTIMATES FOR LOCALLY EUCLIDEAN MANIFOLDS WITH FRACTAL BOUNDARIES

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0. Introduction

Let $M$ be an incomplete $C^\infty$ Riemannian manifold and let $\partial M$ denote the topological boundary of $M$. Let $H$ be the Dirichlet Laplacian on $M$ and denote by $K(t, x, y)$ the kernel of the heat semigroup $e^{-Ht}$. In this paper we investigate certain spectral properties of the operator $H$ in the case where $M$ belongs in a class of two-dimensional, locally-Euclidean manifolds with rough boundary. More precisely the manifolds under consideration are constructed in an iterative way, as for the snowflake domain for example, but are not embeddable in the plane. Both the geometry and analysis of these manifolds have several unusual features. Among these are the fact that the Hausdorff dimension of the boundary $\partial M$ may be greater than the dimension of $M$ itself - a similar result also holds for the Minkowski dimension - and the failure of the Hardy inequality, for which, however, we obtain a useful substitute.

The properties of $H$ which we consider in this paper are the following:

a) Upper bounds on the heat kernel of the form

$$K(t, x, y) \leq ct^{-1}, \quad x, y \in M, \quad t > 0 \quad (0.1)$$

b) Generalized Hardy inequalities of the form

$$d^{-\lambda} \leq cH, \quad 0 < \lambda \leq 2 \quad (0.2)$$

where $d : M \mapsto (0, \infty)$ is the distance from $\partial M$.

c) Compactness of the resolvent operator and behaviour of the ground state eigenfunction at the boundary.

If $M$ is a domain in $\mathbb{R}^2$ then (0.1) is always true as a result of the monotonicity of the Dirichlet heat kernel. If $M$ is also simply connected then (0.2) is true with $\lambda = 2$ for some $c \leq 16$ [A]. The compactness of
the resolvent and the boundary behaviour in c) depend on $M$ in a much more complicated way. However if $M$ is bounded $H$ has discrete spectrum and if (0.2) holds with $\lambda = 2$ then the ground state vanishes continuously at the boundary. On the other hand under various regularity conditions on the boundary [KS, Ch.II] the ground state can be shown to be at least Hölder continuous at the boundary. (This is the case, in particular, for $(\varepsilon, \delta)$ domains and hence also for the snowflake domain [J,D3].)

The case where $M$ is a manifold equipped with a Riemannian metric that becomes singular at the boundary in some controlled way has also been considered in a number of papers [D3,L,P]. Although the spectral properties of $H$ depend heavily on the rate of growth of the volume, one can nevertheless observe that there is a certain interplay between the three problems considered above. In particular inequality (0.2) with $\lambda = 2$ is usually the key to obtaining an upper bound (0.1) on the heat kernel and, combined with a Harnack inequality, can imply the vanishing of the eigenfunctions.

The results presented in this paper show that there exists a large class of Riemannian manifolds that have an “atypical” behaviour vis-a-vis the aforementioned problems. In section 1 we construct the class of manifolds under consideration (it should be noted that we only consider the simplest construction of this type) and calculate the Hausdorff and Minkowski dimensions of the boundary. We also extend the definition of Minkowski dimension to the case where the volume is infinite. We thus discover that the Hausdorff and Minkowski dimensions coincide and their value covers the whole range $(1, +\infty)$. In section 2 we show that a bound of the form (0.1) holds on $M$ for all $0 \leq t \leq 1$, even though, as we show in section 3, inequality (0.2) fails when $\lambda = 2$. However (0.2) holds for all $0 < \lambda < 2$ and is enough to guarantee the discreteness of the spectrum of $H$. Finally in section 4 we show that the ground state eigenfunction $\varphi$ of $H$, and therefore also the heat kernel, cannot decay as fast as any power of the function $d(x)$ as $x \to \partial M$. We also show that $\varphi$ vanishes continuously as $x \to \partial M$ in a certain tangential way but a complete description of the boundary behaviour of $\varphi$ remains open.

1. A Class of 2-dimensional Manifolds With Fractal Boundary

Let $\frac{1}{2} < \alpha \leq 1$ and let $I$ be the set of indices

$$I = \{(n, m) : n \in \mathbb{Z}^+ \setminus \{0\}, 1 \leq m \leq 3.2^n - 1\} \cup \{0\}.$$  

Let $\{T_i\}_{i \in I}$ be a family of triangles in $\mathbb{R}^2$ such that $T_{(n,m)}$ is an isosceles triangle of base length $\alpha^{n-1}$ and side length $\alpha^n$ and $T_0$ is an equilateral