DIVERGENCE IN 3-MANIFOLD GROUPS

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Abstract

The divergence of the fundamental group of compact irreducible 3-manifolds satisfying Thurston's geometrization conjecture is calculated. For every closed Haken 3-manifold group, the divergence is either linear, quadratic or exponential, where quadratic divergence occurs precisely for graph manifolds and exponential divergence occurs when a geometric piece has hyperbolic geometry. An example is given of a closed 3-manifold $N$ with a Riemannian metric of nonpositive curvature such that the divergence is quadratic and such that there are two geodesic rays in the universal cover $\tilde{N}$ whose divergence is precisely quadratic, settling in the negative a question of Gromov's.

1. Introduction

This article is a continuation of [G1]. We have attempted to summarize the results of that earlier article so that the two papers may be read independently.

In [G1] we introduced a quasi-isometry invariant family of functions to measure, roughly speaking, the maximum rate at which geodesic rays diverge in the Cayley graph of a finitely generated group $G$. If $G$ is the fundamental group of a finite CAT(0)-complex $K$, meaning that the universal cover $\tilde{K}$ admits a CAT(0)-metric ([Gr1],[GhH]) such that the deck transformations of $G$ act as isometries, then the divergence of $G$ can be described simply as follows (we refer the reader to [GhH] for the definition of the CAT(0) condition and its properties, especially the article of W. Ballmann). Choose a base point $x_0 \in \tilde{K}$ and let $S_r$ be the sphere of radius $r$ centered at $x_0$ in the CAT(0)-metric. The divergence is the largest diameter $f(r)$ of a connected component of $S_r$, viewed as a function of $r$.

It is shown in [G1] how to modify this definition in the setting of finitely generated groups to get a quasi-isometry invariant; a consequence of the

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definition is that the function \( f(r) \) considered up to an equivalence relation \( f \sim g \) is quasi-isometry invariant; here we write \( f \preceq g \) if there are positive constants \( A, B, C, D, E \) such that \( f(r) \preceq A g(Br + c) + Dr + E \) for all \( r \) and \( f \sim g \) if both \( f \preceq g \) and \( g \preceq f \). Thus it is meaningful to say that the divergence is quadratic, polynomial, or exponential.

In this note we shall calculate the divergence of the fundamental groups of compact irreducible 3-manifolds for which Thurston's geometrization conjecture is valid. For our purposes, a geometric piece is a compact 3-manifold \( P \) with nonempty boundary each of whose boundary components is a torus and such that the interior of \( P \) is either a cusped hyperbolic manifold or is Seifert fibred ([Sc]). In the former case, \( \hat{P} \) admits a complete Riemannian metric of constant negative curvature and finite volume so has \( H^3 \) geometry, and in the latter case \( \hat{P} \) has \( H^2 \times R \) geometry.

We say that the compact connected 3-manifold \( M \) is a **Thurston manifold** if there are a finite number of geometric pieces \( P_1, P_2, \ldots, P_m \) as above such that \( M \) is obtained by gluing certain toric boundary components of their disjoint union in pairs (it is allowed to glue together two boundary components of the same geometric piece). Thus \( M \) has a nonempty boundary iff at least one toric boundary component of the union is left unpaired. We call the images of the boundary components of the pieces in \( M \) the **canonical tori** of \( M \). Note that it follows from our definition that each boundary component of a Thurston manifold is incompressible; thus we have excluded the case of a solid torus, which is not a Thurston manifold although it is Haken.

There are more intrinsic ways of defining the notion of Thurston manifold which allow Klein bottle boundary components as well.\(^1\) However the orientable double covers of such manifolds satisfy our definition of Thurston manifold and the divergence is a quasi-isometry invariant, so there is no loss in taking our definition. We can now state our main results.

**THEOREM 1.** If \( M \) is a Thurston manifold with nonempty boundary and geometric pieces \( P_1, P_2, \ldots, P_m \), then
\[
\text{divergence}(\pi_1(P_i)) \preceq \text{divergence}(\pi_1(M)), \quad 1 \leq i \leq m.
\]

**THEOREM 2.** If all geometric pieces of the Thurston manifold \( M \) with nonempty boundary are Seifert fibred, then
\[
\text{divergence}(\pi_1(M)) \preceq x^2.
\]

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\(^1\)The orientable manifold \( M \) is a Thurston manifold if \( M \) is a compact connected 3-manifold with \( \partial M \neq \emptyset \) and such that \( \partial M \) consists of tori and \( M \) is not a solid torus. It follows that each component of \( \partial M \) is incompressible in \( M \), since we have omitted the single exception of the solid torus. These manifolds are Haken, and hence Thurston's geometrization theorem applies to them.