Geometric And Functional Analysis 1016-443X/94/0600718-31$1.50+0.20/0
Vol. 4, No. 6 (1994)

SYMPLECTIC HOMOLOGY VIA GENERATING FUNCTIONS

LISA TRAYNOR

1. Introduction

Exciting discoveries in the symplectic topology of $(\mathbf{R}^{2n}, \omega_0)$ have been made recently via a technique that combines Hofer's capacity theory and Floer's homology theory into a homology theory with a real filtration ([FH], [FHWy]). The theories of Floer and Hofer naturally fit together since their constructions are both based on studying the action functional on the loop space of a symplectic manifold.

In [V], Viterbo made important progress in capacity theory by viewing the action functional on the loop space as an infinite dimensional limit of generating functions. By means of generating functions, the set of lagrangian submanifolds of a cotangent bundle that can be "defined" by functions is enlarged by considering functions on a vector bundle. Due to results of Sikorav and Laudenbach and of Viterbo, a large class of lagrangians admit "unique" generating functions (§3.3). Viterbo's capacities for lagrangian submanifolds are critical values of the associated generating function. In this manner, capacities are defined for open subsets of $(\mathbf{R}^{2n}, \omega_0)$ via the lagrangians associated to compactly supported symplectomorphisms.

Eliashberg outlined how this work of Viterbo could be extended to an alternate construction of symplectic homology for open subsets of $(\mathbf{R}^{2n}, \omega_0)$ ([El]). Details of this construction are given in §2–§5. In essence, symplectic homology groups for a symplectomorphism h, denoted by $G_*^{(a,b]}(h)$, are the relative homology groups $H_*(E^b, E^a)$, $E^b := \{x \in E : S(x) \leq b\}$, where $S : E \to \mathbf{R}$ is the generating function for the lagrangian associated to h, Definition 3.5. Symplectic homology groups for an open set $U \subset \mathbf{R}^{2n}$ are constructed via a limit of homology groups associated to symplectomorphisms supported on U. These groups are invariants of U under the action

Research at Mathematical Sciences Research Institute supported by an MSRI postdoctoral fellowship. Research at Stanford supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship. Research at Centre Emile Borel, Institut Henri Poincaré, Paris funded by Ministère des Affaires Etrangères, France.

of symplectic diffeomorphisms of $\mathbf{R}^{2n}$. Henceforth, Hofer and Floer's construction will be referred to as F-H homology and this generating function construction as gf-homology.

As opposed to the infinite dimensional analysis behind the construction of F-H homology, the analysis for gf-homology is finite dimensional and elementary. The Conley-Zehnder index and pseudo-holomorphic curves present in the F-H homology theory are replaced by the Morse index of the generating function and the natural boundary operator in a finite dimensional manifold.

It is important to point out that the F-H theory extends to more general symplectic manifolds, [CiFH], while the following is a construction only for open subsets of $(\mathbf{R}^{2n}, \omega_0)$. This technique has potential to be generalized using, for example, ideas of Givental in [G]. In §6, it is shown that many important functorial properties satisfied by F-H homology for $(\mathbf{R}^{2n}, \omega_0)$ are also satisfied by gf-homology. For example, both homology theories have an isotopy invariance property, Theorem 6.8. These functorial properties are central in applications of symplectic homology. Many of the results in [FHWy] can be reproved via gf-homology. A difference between the homology theories is that gf-homology is based on symplectomorphisms defined by compactly supported hamiltonians while the F-H homology is based on hamiltonians that are quadratic at infinity. Hamiltonians which are quadratic at infinity are well adapted to the study of products while it is natural to study disjoint unions via compactly supported symplectomorphisms, Theorem 6.15.

In §7, the gf-homology groups of an open ellipsoid are calculated. These groups are similar to the F-H homology groups ([FHWy]) in the sense that the groups obtained by both theories detect the actions of closed characteristics in the boundary of the ellipsoid. As an illustration, consider

$$E(1,2) := \left\{(x_1^2 + y_1^2) + \tfrac{1}{2}(x_2^2 + y_2^2) < 1\right\} .$$

Examples of gf-homology groups with $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$ coefficients are, for $0 < \epsilon < \pi$,

$$G_*^{(\pi-\epsilon,\pi+\epsilon]}(E(1,2)) = \begin{cases} \mathbf{Z}_2 , & * = 4,5 \\ 0 , & \text{otherwise} ; \end{cases}$$
$$G_*^{(2\pi-\epsilon,2\pi+\epsilon]}(E(1,2)) = \begin{cases} \mathbf{Z}_2 , & * = 6,9 \\ 0 , & \text{otherwise} . \end{cases}$$

The F-H homology groups for a small interval containing π, 2π equal $\mathbf{Z}_2$ only for the indices $* = 3,4$, $* = 5,8$, respectively. For the interval $(-1, \pi + \epsilon]$, the gf-homology groups are non-zero at $* = 0,4,5$ but the corresponding