A Non Locally Convex Example

By

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Following existing terminology, we say that a topological linear space is nearly exotic if it admits no non-trivial continuous linear functionals. In [1] Klee asked whether, if $E$ is an infinite-dimensional nearly exotic topological linear space, $E$ must have a countably infinite dimensional subspace which is nearly exotic (in the relative topology). In [3] we raised the question whether, if $E$ is an infinite-dimensional nearly exotic space, $E$ must have a nearly exotic subspace of dimension less than $\min(\alpha, \beta)$, where $\alpha$ is the minimum dimension of a dense linear subspace of $E$ and $\beta$ is the minimum cardinality of a neighborhood base at the origin for $E$. Our purpose in this note is to answer these questions negatively with an example:

**Theorem.** There is a Hausdorff nearly exotic topological linear space $E$ of algebraic dimension $\aleph_1$ which has the property that if $F$ is any $\aleph_0$-dimensional subspace of $E$ with the relative topology, $F$ admits a separating family of continuous linear functionals.

This example is a sharpening of the example in 1.9 of [3], since we obtain that $E$ is nearly exotic rather than simply that the continuous linear functionals on $E$ fail to separate the points of $E$. It is also basically simpler than the one in [3], since the construction in lemmas 1.11 and 1.12 of that example is not needed here.

If $A$ is a non-empty interval in $[0, 1]$ (not reducing to a point), we define $S(A)$ to be the space of real-valued measurable functions defined on the interval $[0, 1]$ which vanish outside of $A$; two functions are identified if they agree except on a set of Lebesgue measure zero. The convergence in $S(A)$ is defined by means of the metric

$$d(f, g) = \int_0^1 \frac{|f - g|}{1 + |f - g|} d\lambda,$$

where $\lambda$ is Lebesgue measure. With this topology, $S(A)$ is a Hausdorff linear space. (See, for example, [2].)

Let $A = [a, b], A' = [a', b']$, where $a < b, a' < b'$. Then the map

$$x \mapsto a + \left(\frac{x - a'}{b' - a'}\right)(b - a)$$

is a homeomorphism of $S(A)$ onto $S(A')$. This is a sharpening of the example in 1.9 of [3].
of \( A' \) onto \( A \) induces a 1-1 linear map \( \Phi_{A,A'} \) of \( S(A) \) into \( S(A') \) defined by

\[
\Phi_{A,A'}(f) = f \circ \varphi.
\]

The construction of the example rests heavily on the properties of the functions \( \{f_n^A\}_{n=0}^\infty \) in \( S(A) \), defined as follows: if \( A = [a, b] \), set \( f_0^A = \chi_{[a, b]} \), and if \( n \geq 1 \), write \( n \) uniquely as \( 2^i + j, 0 \leq i, 0 \leq j < 2^i \); then set

\[
f_n^A = \chi_{\left[a + \frac{j}{2^i+1}(b-a), a + \frac{j+1}{2^i+1}(b-a)\right]}.
\]

(\( \chi_\gamma \) denotes the characteristic function of the set \( \gamma \)). A dyadic rational point with respect to the interval \( A \) will be any point of the form \( a + \frac{k(b-a)}{2^i} \), where \( 0 \leq i, 0 \leq k \leq 2^i \).

**Lemma 1.** The sequence \( \{f_n^A\} \) is linearly independent in \( S(A) \) and its linear span contains the set of all characteristic functions of intervals whose endpoints are dyadic rational points with respect to \( A \).

**Proof.** Letting \( I \) be the unit interval \([0, 1]\) and making use of the definition of \( \{f_n^A\} \) and the map \( \Phi_{I,A} \), we see that it is enough to prove the assertion in the case that \( A = I \). The proof of the first assertion can be found in 1.9 of [3]. To prove the second assertion, observe that

\[
\chi_{[\frac{1}{2}, 1]} = f_0^I - f_1^I.
\]

Now if \( 0 \leq k < 2^{i+1} \), \( \chi_{\left[k \frac{2^i+1}{2^i+1}, \frac{k+1}{2^i+1}\right]} \) already appears in the sequence \( \{f_n^I\} \) if \( k \) is even; if \( k = 2j + 1 \), then

\[
\chi_{\left[j \frac{2^i+1}{2^i+1}, \frac{j+1}{2^i+1}\right]} = \chi_{\left[j \frac{2^i+1}{2^i+1}, \frac{2}{2^i+1}\right]} - f_{2^i+j}^I.
\]

Thus, induction on the integer \( i \) completes the proof.

For later use, we note the following consequences of (2) and the remarks preceding: for each positive integer \( n \) and each \( k, 0 \leq k < 2^n \), the function \( \chi_{\left[k \frac{2^n}{2^n}, \frac{k+1}{2^n}\right]} \) is in the linear span of \( \{f_j^I\} \), by Lemma 1, and plainly

\[
f_0^I = \frac{1}{2^n} \sum_{k=0}^{2^n-1} 2^n \chi_{\left[k \frac{2^n}{2^n}, \frac{k+1}{2^n}\right]}.
\]

Applying \( \Phi_{I,A} \) we obtain

\[
f_0^A = \frac{1}{2^n} \sum_{k=0}^{2^n-1} 2^n \Phi_{I,A}\left(\chi_{\left[k \frac{2^n}{2^n}, \frac{k+1}{2^n}\right]}\right).
\]

When each \( 2^n \chi_{\left[k \frac{2^n}{2^n}, \frac{k+1}{2^n}\right]} \) is expressed in terms of the functions \( f_j^I \), (3) becomes an identity in the linearly independent elements \( f_j^I \) and (4) is the corresponding identity in the functions \( f_j^A \).

Now suppose that \( \mathcal{A} \) is an index set and that for each \( \alpha \) in \( \mathcal{A} \), \( A_\alpha \) is a non-empty interval in \([0, 1]\), not reducing to a point. Let \( \{f_n\}_{n=0}^\infty \) be the sequence in \( \prod_{\alpha \in \mathcal{A}} S(A_\alpha) \).