Note on the Smoothness of Integral Means

To ALEXANDER OSTROWSKI for his 60th anniversary

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1. Introduction

If a function \( f(x) \) is continuous, its mean \( M_h(x) \) over the interval \((x, x+h)\) has a continuous derivative. Recently, L. D. THOMPSON [1] has shown among other results involving higher derivatives, that conversely, the continuity of \( M_h \), together with one other condition, implies that \( f \) is continuous. The additional condition is that \( f \) should be \emph{means continuous}, or that \( M_h \) should approach \( f \) as \( h \) tends to zero. It is pointed out in [1] that some additional condition is required for the converse to hold, as the example of a function \( f \) equal almost everywhere to a continuous function clearly shows.

Now the mean continuity of \( f \) implies by itself a considerable amount about the continuity of \( f \); it says that \( f \) is the derivative everywhere of its integral and thus has the \emph{Darboux} property of taking on all intermediate values and is continuous on a set of second category-all this without reference to \( M_h(x) \). Thus it would appear of some interest to obtain a converse theorem without this additional hypothesis. In the following we shall show that in a sense, the extra condition can be dispensed with entirely and that the above type of counter example is the only exception to a strict converse. To be precise, we prove the following theorem.

\textbf{Theorem 1:} If \( f \) is summable and \( M_h \) has a continuous derivative, then \( f \) is equal almost everywhere to a continuous function.

2. A lemma

The following lemma is somewhat more general than is required for the proof of the theorem.

\textbf{Lemma:} If \( g \) is a measurable function satisfying for each \( h \neq 0 \) the condition that \( g(x+h) - g(x) \) be continuous, then \( g \) is continuous.

Set \( g(x+h) - g(x) = p_h(x) \), where \( p_h \) is continuous. It is clear that if \( g \) is discontinuous at one point, it is so at all points and the nature of the discontinuity at all points is the same. Suppose that \( g \) is discontinuous at \( x_0 \). No loss of generality
is suffered by assuming that \( \lim_{t \to x} g(t) - g(x_0) > \delta > 0 \), and it follows immediately that for any \( x \), \( \lim_{t \to x} g(t) - g(x) > \delta \). Thus, given \( \varepsilon > 0 \), there is a point \( x_1 \) with \( |x_1 - x_0| < \varepsilon/2 \) and \( g(x_1) > g(x_0) + \delta \); similarly there is a point \( x_2 \) with \( |x_2 - x_1| < \varepsilon/4 \) and \( g(x_2) > g(x_1) + \delta > g(x_0) + 2\delta \), etc. In this way we see that \( g \) is unbounded above in the \( \varepsilon \)-neighborhood of \( x_0 \). Let \( k_i \) be chosen so that \( k_i \to 0 \) as \( i \to \infty \) and \( g(x_0 + k_i) \to \infty \). It follows that for any \( x \), \( g(x + k_i) \to \infty \). Set
\[
G(x) = \frac{g(x)}{1 + |g(x)|}.
\]

\( G \) is bounded and measurable, \( |G| < 1 \), but \( G(x + k_i) \to 1 \) for all \( x \). Thus for any \( a, b \),
\[
\int_a^b |G(x + k_i) - G(x)| \, dx \to \int_a^b 1 - G(x) \, dx = 0.
\]

However, for any summable function \( G \),
\[
\lim_{k \to 0} \int_a^b |G(x + k) - G(x)| \, dx = 0.
\]

This involves a contradiction and the lemma is proved.

It is perhaps of interest to note that without the hypothesis of measurability, the lemma is not true. In fact, let \( g \) be any discontinuous solution of the functional equation \( g(x + y) = g(x) + g(y) \). (For a discussion of these functions see [2].) Then \( g(x + h) - g(x) = g(x) + g(h) - g(x) = g(h) = \text{constant} \).

3. Proof of theorem 1

Let \( f \) be summable and \( M^*_h \) continuous for all \( h \neq 0 \). If \( F(x) = \int_a^x f(t) \, dt \), then
\[
h \, M^*_h(x) = F(x + h) - F(x).
\]

If \( F'(x) \) exists, then \( F'(x + h) \) exists; that is, if \( F' \) exists at one point it exists at all. We know that \( F' \) does exist and equal \( f \) almost everywhere. Thus \( F' \) exists at all points and satisfies the equation,
\[
F'(x + h) - F'(x) = h \, M^*_h(x).
\]

By the lemma, \( F' \) is continuous, and since \( f = F' \) almost everywhere, the statement of the theorem follows.

4. Extension to higher derivatives

Let \( M_h^{(n+1)} \) be continuous, \( n \geq 0 \). In particular, \( M_h \) is continuous and by theorem 1, \( f = f_1 + f_2 \), where \( f_1 \) is continuous and \( f_2 = 0 \) almost everywhere. The means