Möbius Inversion in Lattices

By

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1. Introduction. In the development of computational techniques for combinatorial theory, attention has lately centered on Rota's theory of Möbius inversion [6]. The main theorem of Rota's paper, concerning the computation of the Möbius invariant across a Galois connection, is a prerequisite to the use of lattice-theoretic methods in combinatorics.

By suitably combining Rota's main theorem with a discrete analogue of integration-by-parts, we here obtain a perfectly general formulation of Möbius inversion across a Galois connection (theorem 3, below).

As immediate applications of this theory, we obtain a number of interesting computational results concerning finite lattices (section 3, 4) and combinatorial geometries (section 5).

2. Möbius Inversion across a Galois Connection. We begin with a restatement and a simplified proof of Rota's main theorem. The proof turns on the essential fact that for any (locally finite) ordered set $Q$ with least element $0$, the recursion

$$
\sum_{y \in Q} a(y) \zeta(y, z) = 0 \text{ for } z \neq 0
$$

has the unique solution $a(y) = 0$ with initial condition $a(0) = 0$, and has the unique solution $a(y) = \mu_Q(0, y)$ with initial condition $a(0) = 1$. Recall that the zeta function $\zeta(y, z)$ has value 1 if $y \leq z$, and has value 0 otherwise.

Theorem 1. If $J$ is a closure operator on a finite lattice $P$, and $Q = P/J$ is the quotient lattice, consisting of the $J$-closed elements of $P$, then for all elements $x \in P$ and elements $y$ closed in $P$, $x \leq y$, the sum

$$
\sum_{t; x \leq t \leq J(t) = y} \mu(x, t)
$$

has value $\mu_Q(x, y)$ if $x$ is closed, and has value 0 otherwise.

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Proof. Note that the theorem may be rewritten in the form

\[ \delta(x, J(x)) \mu_Q(J(x), y) = \sum_{t \in P} \mu(x, t) \delta(J(t), y). \]

Without loss of generality, we assume \( x = 0 \) in \( P \). For each element \( y \in Q \), let

\[ a(y) = \sum_{t \in P} \mu(0, t) \delta(J(t), y) \zeta(t, y). \]

Then

\[ a(y) \zeta(y, z) = \sum_{t, y \in P} \mu(0, t) \delta(J(t), y) \zeta(y, z) = \sum_{t \in P} \mu(0, t) \zeta(t, z) = \delta_P(0, z). \]

If \( 0 < J(0) \), \( \delta_P(0, z) = 0 \) for all \( z \in Q \), and \( a(y) = 0 \) for all \( y \in Q \). If \( 0 = J(0) \), \( \delta_P(0, z) = 1 \) for \( z = 0 \), and \( a(y) = \mu_Q(0, y) \).

Given a function \( f \) from a finite lattice \( P \) into a ring with unit, associate the difference operators \( D, E \)

lower difference \[ Df(x) = \sum_{y \leq x} \mu(y, x) \]
upper difference \[ Ef(x) = \sum_{y : x \leq y} \mu(x, y) f(y). \]

Theorem 2 (Analogue of integration by parts). If \( f, g \) are functions from a finite lattice \( P \) into a ring, then

\[ \sum_{x \in P} Df(x) g(x) = \sum_{x \in P} f(x) Eg(x). \]

Proof. Both are equal to \( \sum_{x, y} f(x) \mu(x, y) g(y) \).

It is interesting to compare the proof of theorem 2 with the argument that cycles and coboundaries in a graph are orthogonal to one another. For each vertex \( p \) and edge \( x \), let

\[ \epsilon(p, x) = \begin{cases} \pm 1 & \text{if } p \text{ is the head of } x, \\ -1 & \text{if } p \text{ is the tail of } x, \\ 0 & \text{otherwise}. \end{cases} \]

Boundary and coboundary operators are defined by

\[ \partial f(p) = \sum_{x \in P} \epsilon(p, x) f(x) \]
\[ \delta g(x) = \sum_{p} g(p) \epsilon(p, x) \]

If \( f \) is a 1-cycle \( (\partial f = 0) \) and \( h \) is a 1-coboundary \( (h = \delta g) \), then

\[ \sum_x f(x) h(x) = \sum_x f(x) \epsilon(p, x) g(p) = \sum_p \partial f(p) g(p) = \sum_p 0 g(p) = 0. \]

If \( \sigma : P \rightarrow L \) is a supremum-homomorphism from a complete lattice \( P \) into a complete lattice \( L \), then \( \sigma^4 : L \rightarrow P \) is an infimum-homomorphism, defined by

\[ \sigma^4(y) = \sup \{ x; \sigma(x) \leq y \}. \]

The pair \( \sigma, \sigma^4 \) is a Galois connection, in the sense that \( P/\sigma^4(\sigma) \) is isomorphic to \( L/\sigma(\sigma^4) \), where \( \sigma^4(\sigma) \) is a closure operator on \( P \) and \( \sigma(\sigma^4) \) is a coclosure operator on \( L \). All Galois connections between complete lattices arise in this fashion. In the special case where \( \sigma \) is onto \( L \), \( P/\sigma^4(\sigma) \cong L \).