Translations in finite Möbius planes

By

Judita Cofman

1. An automorphism \( \tau \) of a Möbius plane \( \mathcal{M} \) is called a translation if it is the identity or if it has exactly one fixed point \( T \) and fixes all circles of a tangent bundle \( t \) through \( T \). The point \( T \) is called the centre and the tangent bundle \( t \) the co-centre of \( \tau \).

The translations of the Möbius planes play a role similar to that of the elations in the theory of projective planes. In this note this similarity will be illustrated by proving, for finite Möbius planes, an analogue of the following result of Gleason [4] on finite projective planes: If \( \pi \) is a finite projective plane with a collineation group containing for every incident point-line pair \( P, g \) a non-identical elation with centre \( P \) and axis \( g \) then \( \pi \) is desarguesian. In the case of finite Möbius planes the following can be proved:

Theorem. If \( \mathcal{M} \) is a finite Möbius plane with an automorphism group containing for each incident point-tangent bundle pair \( P, t \) a non-identical translation with centre \( P \) and co-centre \( t \) then \( \mathcal{M} \) is miquelian.

For the definition and basic properties of a Möbius plane see for instance Dembowski [2].

2. Proof of the Theorem. Let \( \mathcal{M} \) be a finite Möbius plane of order \( n \) with an automorphism group \( A \) which contains for every incident point-tangent bundle pair \( P, t \) a non-identical translation with centre \( P \) and co-centre \( t \).

Throughout the following investigations the subgroup of \( A \) consisting of all translations with centre \( P \) and co-centre \( t \) will be denoted by \( A_{P,t} \) or by \( A_{P,t} \) where \( t \) is a circle of \( t \).

Take an arbitrary point \( A \) of \( \mathcal{M} \). The translations of \( A \) with centre \( A \) form a group \( A_A \). Denote by \( \mathcal{M}(A) \) the affine subplane of \( \mathcal{M} \) at \( A \). The group \( A_A \) induces a translation group \( \bar{A}_A \) in \( \mathcal{M}(A) \) of order \( | \bar{A}_A | = | A_A | \). Since the tangent bundles of \( \mathcal{M} \) through \( A \) represent the improper points of \( \mathcal{M}(A) \) it follows that in \( \mathcal{M}(A) \) every improper point is centre of a nontrivial translation. Thus \( | \bar{A}_A | > n \) and, according to Dembowski [3]:

1. \( n \) is a prime power \( p^a \), where \( p \) is the order of any non-trivial translation in \( \mathcal{M}(A) \), and

2. for at least one improper point \( M_\infty \) of \( \mathcal{M}(A) \) the group \( \bar{A}_A \) is transitive on the affine lines of \( \mathcal{M}(A) \) through \( M_\infty \).
(1) implies that all non-identical translations of $\mathcal{M}$ are of the same prime order $p$. Consequently $A$ contains for any point $P$ of $\mathcal{M}$ an automorphism of order $p$ fixing $P$ but no other point of $\mathcal{M}$; hence, according to Gleason [4],

(3) $A$ is transitive on the points of $\mathcal{M}$.

The stabilizer $A_k$ of any circle $k$ of $\mathcal{M}$ contains a translation of order $p$ with centre $X$ and co-centre $z \not\in k$ for every point $X \in k$; therefore, by Gleason [4], it follows that $A_k$ is transitive on the points of $k$. Hence:

(4) $|A_{X,k}| = |A_{Y,k}|$ for any two points $X, Y$ on $k$.

Let $k$ and $k'$ be two circles through a point $K$ such that there is an automorphism $\alpha$ of $A$ mapping $k$ onto $k'$. Denote by $K'$ the image of $K$ under $\alpha$. Clearly, $|A_{K,k}| = |A_{K',k'}|$ and according to (4), $|A_{K',k'}| = |A_{K,k}|$. This implies:

(5) If two circles $k$ and $k'$ through a point $K$ are in the same orbit of $A$ then $|A_{K,k}| = |A_{K,k'}|$.

In view of (2) the affine plane $\mathcal{M}(A)$ has an improper point $M_\infty$ such that $\mathcal{A}$ is transitive on the affine lines of $\mathcal{M}(A)$ through $M_\infty$. Denote by $m$ the tangent bundle through $A$ which corresponds to the improper point $M_\infty$ in $\mathcal{M}(A)$. Then the $n$ circles of $m$ constitute an orbit of $A_A$, hence $|A_{A,m}| = |A_A|/n$. Denote by $p^b$ the order of $A_A$; then $|A_{A,m}| = p^{b-a}$.

Take a circle $m \in m$ and denote by $\mathcal{O}$ the circle orbit of $A$ containing $m$. Let $\mathcal{O}_A$ be the subset of $\mathcal{O}$ which consists of the circles through $A$. According to (5), if $k$ is any circle of $\mathcal{O}_A$ then $|A_{A,m}| = p^{b-a}$; hence $A_A$ maps $k$ onto $|A_A|/|A_{A,m}| = n$ distinct circles of the tangent bundle determined by $A$ and $k$. Thus

(6) If $\mathcal{O}_A$ contains a circle $k$, then all circles of the tangent bundle, determined by $A$ and $k$ are in $\mathcal{O}_A$.

This implies, if $|\mathcal{O}_A|$ denotes the number of distinct elements in $\mathcal{O}_A$:

(7) $|\mathcal{O}_A| \equiv 0 \mod n$.

Consider now the following incidence structure $\mathcal{S}$: the points of $\mathcal{S}$ are the points of $\mathcal{M}$, the blocks of $\mathcal{S}$ are the circles of $\mathcal{O}$ and incidence is the same as in $\mathcal{M}$. There are $n^2 + 1$ points in $\mathcal{S}$ and any block of $\mathcal{S}$ is incident with $n + 1$ points. According to (3) each point is incident with the same number of blocks which is equal to $|\mathcal{O}_A|$. Denote by $|\mathcal{O}|$ the number of blocks in $\mathcal{S}$; counting the incident point-block pairs of $\mathcal{S}$ in two different ways we obtain

(8) $(n^2 + 1) \cdot |\mathcal{O}_A| = (n + 1) \cdot |\mathcal{O}|$.

Since $|\mathcal{O}_A| \equiv 0 \mod n$ and since the greatest common divisor of $(n^2 + 1)$ and $n + 1$ is 1 if $n$ is even and 2 if $n$ is odd, it follows that for $n$ even $|\mathcal{O}|$ is a multiple of $(n^2 + 1)$ and for $n$ odd $|\mathcal{O}|$ is a multiple of $(n^2 + 1)/2$.

However the total number of circles in $\mathcal{M}$ is $(n^2 + 1)$. Hence either

(a): $|\mathcal{O}| = (n^2 + 1)\cdot n$ or (b): $|\mathcal{O}| = (n^2 + 1)/2$. 
