EDDY CURRENT ANALOGY OF TORSION PROBLEM

Wang Zhen-min (王国振) Zhang Ke-xue (张可雪)
(Jiangsu Institute of Technology, Jiangsu)
(Received Aug. 10, 1985 Communicated by Chien Wei-zang)

Abstract

In this paper, an eddy current analogy and a brief sketch of required equipment are presented. Values of torsional rigidity and shearing stresses of a prismatic bar under free torsion can be obtained experimentally to a high degree of accuracy in an instant with this equipment whether the cross-section is bounded by a single boundary or multi-connected boundaries. The error is less than two per cent generally, as shown in Table 3. This new analogy can be used extensively to solve various physical problems expressed by Poisson's (or Laplace's) equation with constant boundary conditions.

In Theory of Elasticity, it is difficult to find analytical solutions for some prismatic shafts with special cross-section under free torsion. Only a few shafts with simple cross-section have been solved. To solve general cross-sectional problems, we have to use approximate methods or approximate calculations, such as the finite element method. Both of these methods required long and tedious mathematical calculations. The alternative way is to solve the problem by using analogy methods. These analogies can be classified into three types: namely, membrane analogy, electrical analogy and fluid flow analogy. Some useful results of practical problems were obtained by the membrane analogy method. But it is rather difficult to make accurate measurements of the small transverse displacements of the membrane. Furthermore, it needs special equipment to fulfil the requirement of multi-connected cross-section. By the new method, however, these troublesome procedures do not exist any more. The eddy current analogy overcomes all the above-mentioned shortcomings. Once a piece of thin conductor plate, with the shape of desired cross-section, is put into the gap of the instrument, the values of shear stresses at certain points and the torsional rigidity will be expressed digitally in a moment.

I. Theory of Eddy Current Analogy for Free Torsion

1. Free torsion problem of elastic prismatic shaft

An elastic prismatic shaft under the action of torques at both ends is shown in Fig 1. The cross-sectional boundary \( \Gamma \) of the shaft consists of the external boundary \( l_0 \) and several internal boundaries \( m_i (i = 1, 2, \ldots, n) \) (section with a single boundary is the special case, \( n = 0 \)), as shown in Fig. 2. The area bounded by \( l_0 \) and \( m_i (i = 1, 2, \ldots, n) \) is denoted by \( A_i \) (not including the areas of hollow spaces), while areas bounded by \( m_i \) are denoted by \( A_i (i = 1, 2, \ldots, n) \). The well-known Prandtl stress function \( \phi(x, y) \) of the section
should satisfy the following three equations:

\[ \nabla^2 \phi(x, y) = -2 \quad ((x, y) \in \mathcal{A}_e) \]  
\[ \frac{d\phi(x, y)}{ds} = 0 \quad ((x, y) \in \Gamma) \]  
\[ \oint_{m_i} \frac{\partial\phi}{\partial y} \, dx - \frac{\partial\phi}{\partial x} \, dy = 2A_i \quad (i=1, 2, \ldots, n) \]  

in which \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplace operator; \( s \) denotes the path along any boundary. For single connected boundary, Eq.(1.3) is longer existent.

From Eq.(1.2) we know that along all the external and internal boundaries, \( \phi(x, y) \) are constants. One of them is independent and can be freely set: generally we let

\[ \phi(x, y) = 0 \quad ((x, y) \in l_o) \]  

and take

\[ \phi(x, y) = K_i \quad ((x, y) \in m_i, \ i=1, 2, \ldots, n) \]  

Shearing stresses on cross-section and rigidity of shaft can be obtained by the following equations:

\[ \tau_{zx} = G\theta \frac{\partial\phi}{\partial y}, \quad \tau_{zy} = -G\theta \frac{\partial\phi}{\partial x} \]  
\[ \frac{M}{G\theta} = 2\left(\iint_{\mathcal{A}_o} \phi \, dx \, dy + \sum_{i=1}^{n} K_i A_i \right) \]  

In which \( \tau_{zx}, \tau_{zy} \) are respectively \( x \) and \( y \) components of shearing stress \( \tau \) at any point on the cross-section; \( G \) denotes the shear modulus of rigidity of material; \( \theta \) is the twist of unit length. If the cross-section is single-connected boundary, in Eq.(1.7), the second term in the bracket is no longer existent.

**2. Theory of eddy current analogy**

Now let us consider the electric eddy current problem for a thin conductor plate in a magnetic field. There is a non-magnetic thin conductor plate of unit depth with several holes (as shown in Fig. 3). Here we still use the same notations \( \Gamma, l_o, m_i, A_o, A_i \), and so on as mentioned above.

Let a beam magnetic lines pass through the plate perpendicularly. The magnetic strength