ON THE PROBLEM OF PREVENTING BLOWING-UP AND QUenchING FOR SEMILINEAR HEAT EQUATION

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Abstract

In this paper, the global existence of solutions to the IVP
\[ \Delta u + g(t)f(u) \quad (t > 0), \quad u_{t}|_{t=0} = u_0(x) \]

and the IBVP
\[ u_{t} = \Delta u + g(t, x)f(u) \quad (t > 0, \quad x \in \Omega), \quad u_{t}|_{t=0} = u_{|x=0} = 0 \]
is investigated. As has been done in [6], the introduction of factor \( g(t) \) or \( g(t, x) \) in nonlinear term is to prevent the occurrence of blowing-up or quenching of solutions. It is shown in this paper that most of the restrictions on \( g \) and \( u_0 \) in the theorems of [6] may be cancelled or relaxed, that the smallness of \( g \) is required only for \( t \) large, and that under certain conditions controlling initial state can avoid blowing-up.

I. Introduction

As a result of practical need and theoretical interest, the problem that solutions of semilinear heat equation may blow up or quench has been attracting more and more attentions. In 1963, Kaplan \[11\] for the first time showed that if \( f(u) \) is a positive valued convex function for \( u \) large and satisfies the condition
\[ \int_{0}^{\infty} \frac{du}{f(u)} < \infty \]
then any solution \( u \) of the semilinear parabolic equation
\[ u_{t} = Lu + f(u) \]
where \( L \) is a linear uniformly elliptic operator of second order, must blow up, i.e., \( u \) becomes infinite in finite time when its initial value or initial-boundary value is large enough. Afterwards, Fujita \[12\], Hayakawa \[11\] and others thoroughly investigated the equation
\[ u_{t} = \Delta u + u^a \quad (a > 1) \]
and achieved the following fairly satisfactory results:

If \( a \leq (n + 2)/n \), then, for any nontrivial initial value, the corresponding initial value problem has no nonnegative global solutions, i.e., any nonnegative solution blows up in finite time; if \( a > (n + 2)/n \), then, for nonnegative initial value small enough, the global solutions exist.
There have also been some excellent works concerning more general nonlinear term \( f(u) \), instead of \( u^a \).

On the other hand, in 1975, Kawarada \([\text{I}]\) found that for the initial-boundary value problem

\[
\begin{align*}
  u_t &= u_{xx} + \frac{1}{1-u} : 0 < t < T \leqslant \infty, \quad x \in (0, a) \\
  u(0, x) &= u(t, x) = 0
\end{align*}
\]

there exists a positive number \( a^* \), called "critical value", such that for \( 0 < a < a^* \), we have global existence, and for \( a > a^* \), quenching occurs, i.e., a solution exists and is bounded only in finite time and its derivatives blow up. Acker and Walter \([\text{II}]\) worked on the similar problem for more general nonlinear term \( f(u) \), and showed the close connection between the global existence for a non-stationary problem and the existence for the corresponding stationary problem.

In order to prevent the occurrence of blowing-up or quenching, Chen Ching-yi \([\text{III}]\) introduced an anti-blowing-up or anti-quenching factor into the nonlinear term, i.e., \( g(t) f(u) \) or \( g(t, x) f(u) \) takes the place of \( f(u) \) with \( g(t) \) or \( g(t, x) \) small enough in a sense. This may be an interesting idea. However, in Reference \([\text{IV}]\), the restrictions placed on \( g(t) \), \( g(t, x) \), \( f(u) \) and initial values are too strong and there are some mistakes in the calculation of \( a^* \) for the initial-boundary value problem (1.1).

The purpose of this paper is to improve the results of Reference \([\text{IV}]\). It is shown that most of the restrictions on \( f, g \) and \( u_0 \) in the theorems of \([\text{IV}]\) may be cancelled or relaxed, that the smallness of \( g \) is required only when \( t \) is large enough, and that under certain conditions controlling initial state can avoid blowing-up.

II. The Problem of Anti-blowing-up

Consider the initial value problem for semilinear heat equation:

\[
\begin{align*}
  u_t &= \Delta u + g(t) f(u), \quad t > 0, \quad x \in \mathbb{R}^n \\
  u(0, x) &= u_0(x), \quad x \in \mathbb{R}^n
\end{align*}
\]

**Theorem 1** Let \( f(u) \) and \( g(t) \) be continuously differentiable functions, and \( u_0(x) \) the continuous function. Suppose that

\[
|f(u)| \leqslant |u|^\alpha, \quad \alpha > 1 \tag{2.2}
\]

and

\[
\int_0^\infty |g(t)| \, dt = K < +\infty \tag{2.3}
\]

and

\[
|u_0(x)| \leqslant M \tag{2.4}
\]

Then, the initial value problem (2.1) has bounded, global solutions defined for all \( t > 0 \) and no blowing-up occurs, provided that \( K \) or/and \( M \) is so small that

\[
MK^{\frac{1}{\alpha-1}} \leqslant \left(1 - \frac{1}{\alpha}\right)^{\frac{1}{1-\alpha}} \tag{2.5}
\]

**Proof** Denote \( r = (aK)^{\frac{1}{\alpha-1}} \). By inequality (2.5), \( M \leqslant (1 - \frac{1}{\alpha})^{\frac{1}{1-\alpha}} \). It is well-known that initial value problems

\[
\begin{align*}
  w_t &= \Delta w + r^\alpha|g(t)|, \quad t > 0, \quad x \in \mathbb{R}^n \\
  w(0, x) &= u_0(x), \quad x \in \mathbb{R}^n
\end{align*}
\]


