Comments on "Aggregation of Equivalence Relations"
by P. C. Fishburn and A. Rubinstein

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Abstract: The theorem of the paper "Aggregation of Equivalence Relations," by Fishburn and Rubinstein, states a result already known. This theorem improves a result from Mirkin (1975) and appears as a corollary occurring in Leclerc (1984).


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The paper " Aggregation of Equivalence Relations" (Fishburn and Rubinstein 1986) concerns consensus theory for classifications and an Arrowian oligarchic theorem for equivalence relations: under an independence condition and the unanimity rule, a consensus between equivalence relations can be expressed as an intersection rule.

To obtain this result, the authors define an aggregator as a function $F$ that assigns a single equivalence relation on a given finite set $A$ to each possible $n$-tuple $(R_1, \ldots, R_n)$ of equivalence relations on $A$. Then, following the Arrowian framework (Arrow 1951), they introduce the notion of consistency: An aggregator $F$ is said to be consistent if it satisfies the following two conditions:

\[ C_1: \text{For all } a, b \in A \text{ and all } n\text{-tuples } (R_1, \ldots, R_n) \text{ and } (R'_1, \ldots, R'_n) \text{ of equivalence relations on } A, \text{ if } aR_ib \text{ is equivalent to } aR'_ib, \text{ for } i = 1, \ldots, n, \]

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then \( aRb \) if and only if \( aR'b \), \( R \) (\( R' \)) being defined as: \( F(R_1, \ldots, R_n) (F(R'_1, \ldots, R'_n)) \).

\[ C_2: \text{For all } a, b \in A \text{ and all } n\text{-tuples } (R_1, \ldots, R_n) \text{ of equivalence relations on } A, \text{ if } aR_i b, \text{ for } i = 1, \ldots, n, \text{ then } aRb \text{ and if not } (aR_i b), \text{ for } i = 1, \ldots, n, \text{ then not } (aRb). \]

Lastly the authors introduce the class of conjunctive aggregators. An aggregator \( F \) is said to be conjunctive if there is some nonempty subset \( N \) of \( \{1, \ldots, n\} \) such that for all \( a, b \in A \) and all \( n\)-tuples \( (R_1, \ldots, R_n) \) of equivalence relations on \( A \), \( aRb \) if and only if \( aR_i b \), for every \( i \in N \). In other words, \( F(R_1, \ldots, R_n) = \cap \{R_i : i \in N\} \).

With that material Fishburn and Rubenstein prove the following:

**Theorem.** Suppose \( A \) has at least three elements. Then the set of consistent aggregators equals the set of conjunctive aggregators.

This result has been known in Consensus Theory since 1975. Several survey papers on many topics of this kind are available (see, for example Barthelemy, Leclerc and Monjardet 1984, 1986). I will just point out two facts:

First, a popular result in the current literature on the consensus of classifications and very similar to this theorem was established by Mirkin (1975), with the help of stronger conditions. Mirkin appears to have been the first to apply the Arrowian approach to equivalence relations and, even if his conditions appear somewhat redundant, Mirkin should be credited with the influential ideas in that domain. Moreover, it is easy to see Mirkin’s Theorem within the Fishburn and Rubinstein’s formulation, as was done in Barthelemy et al. (1984).

Second, more recently, a general result involving valued relations has been obtained by Leclerc (1984) and admits many far-reaching consequences. Among them, the theorem of Fishburn and Rubenstein is exactly (up to some minor changes in the terminology) corollary 7.3 in Leclerc’s (p. 54) paper. Leclerc attributes this result explicitly to Mirkin. A full statement from Leclerc’s paper is (with \( X \) changed to \( A \), \( v \) to \( n \), \( \Psi \) to \( F \), \( W \) to \( N \), \( V \) to \( \{1, \ldots, n\} \), \( E \) to \( R \)):

**Corollary 7.3** (Mirkin, 1975, in the formulation of Barthelemy et al., 1984). A consensus function \( F : E^n \rightarrow E \) on equivalence relations on \( A \) is efficient and binary iff there exists \( N \subseteq \{1, \ldots, n\} \) such that, for any \( \eta = (R_1, \ldots, R_n) \in E^n, F(\eta) = \cap \{R_i : i \in N\} \).