ON THE RIEMANNIAN CURVATURE
OF A TWISTOR SPACE

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§ 1. Introduction. The twistor space of an oriented Riemannian 4-manifold $M$ is the 2-sphere bundle $Z$ on $M$ consisting of the unit $(-1)$-eigenvectors of the Hodge star operator acting on $\wedge^2TM$. The 6-manifold $Z$ admits a natural 1-parameter family of pseudo-Riemannian metrics $h_t$, $t \neq 0$. For $t > 0$, these metrics are definite and have been studied by Friedrich and Kurke [4] in connection with the classification of self-dual Einstein 4-manifolds with positive scalar curvature. In [3], Friedrich and Grunewald have given the geometric conditions on $M$ ensuring that $h_t$, $t > 0$, is an Einstein metric. In the case $t < 0$, $h_t$ is indefinite and has been studied by Vitter in [10] where local formulas for the curvature and Ricci forms have been obtained. K. Sekigawa [8] has considered the metrics $h_t$, $t > 0$, on the twistor space of an oriented Riemannian 2n-manifold.

The main purpose of this paper is to give a coordinate-free formula for the sectional curvature of the pseudo-Riemannian manifold $(Z, h_t)$ in terms of the curvature of $M$. This is achieved by means of the O’Neill formulas [6] for Riemannian submersions. As applications we discuss the Ricci curvature of $(Z, h_t)$ and the holomorphic sectional curvatures with respect to the almost complex structures on $Z$ introduced by Atiyah, Hitchin and Singer [1] and Eells and Salamon [2], respectively.

§ 2. Preliminaries. Let $M$ be an oriented Riemannian 4-manifold with metric $g$. Then $g$ induces a metric on the bundle of 2-vectors $\wedge^2TM$ by the formula

$$g(A_1 \wedge A_2, A_3 \wedge A_4) = \frac{1}{2} \det (g(A_i, A_j)).$$

The Riemannian connection of $M$ determines a connection of the vector bundle $\wedge^2TM$ (both denoted by $\nabla$) and the respective curvatures are related by

$$R(A \wedge B)(C \wedge D) = R(A, B)C \wedge D + C \wedge R(A, B)D$$

for $A, B, C, D \in \mathcal{X}(M)$; $\mathcal{X}(M)$ stands for the Lie algebra of smooth vector fields on $M$. (For the curvature tensor $R$ of $M$ we adopt the following

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definition: $R(A, B) = \nabla_{[A,B]} - [\nabla_A, \nabla_B]$. The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(A \wedge B), C \wedge D) = g(R(A, B)C, D)$$

for all $A, B, C, D \in \mathcal{X}(M)$. The Hodge star operator defines an endomorphism $\ast$ of $\Lambda^2 TM$ with $\ast^2 = \text{Id}$. Hence

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM$$

where $\Lambda^2_\pm TM$ are the subbundles of $\Lambda^2 TM$ corresponding to the $(\pm 1)$-eigenvectors of $\ast$. Let $(E_1, E_2, E_3, E_4)$ be a local oriented orthonormal frame of $TM$. Set

$$s_1 = E_1 \wedge E_2 - E_3 \wedge E_4, \quad \bar{s}_1 = E_1 \wedge E_2 + E_3 \wedge E_4,$$

$$s_2 = E_1 \wedge E_3 - E_4 \wedge E_2, \quad \bar{s}_2 = E_1 \wedge E_3 + E_4 \wedge E_2,$$

$$s_3 = E_1 \wedge E_4 - E_2 \wedge E_3, \quad \bar{s}_3 = E_1 \wedge E_4 + E_2 \wedge E_3.$$

Then $(s_1, s_2, s_3)$ (resp. $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$) is a local oriented orthonormal frame of $\Lambda^2_- TM$ (resp. $\Lambda^2_+ TM$). The matrix of $\mathcal{R}$ with respect to the frame $(s_i, s_i)$ of $\Lambda^2 TM$ has the form

$$\mathcal{R} = \begin{bmatrix} A & B \\ t_B & C \end{bmatrix}$$

where the $3 \times 3$ matrices $A$ and $C$ are symmetric and have equal traces. Let $\mathcal{B}$, $\mathcal{W}_+$ and $\mathcal{W}_-$ be the endomorphisms of $\Lambda^2 TM$ with matrices

$$\mathcal{B} = \begin{bmatrix} 0 & B \\ t_B & 0 \end{bmatrix}, \quad \mathcal{W}_+ = \begin{bmatrix} A - \lambda I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{W}_- = \begin{bmatrix} 0 & 0 \\ 0 & C - \lambda I \end{bmatrix}$$

where $\lambda = \frac{1}{3} \text{Trace } C$ and $I$ is the unit $3 \times 3$ matrix. Then $\mathcal{R} = \lambda \text{Id} + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_-$ is the irreducible decomposition of $\mathcal{R}$ under the action of $SO(4)$ found by Singer and Thorpe [9]. Note that $\lambda = 1/6$ scalar curvature; $\lambda \text{Id} + \mathcal{B}$ and $\mathcal{W}_+ + \mathcal{W}_-$ represent the Ricci tensor and the Weyl conformal tensor, respectively. The manifold $M$ is called self-dual (anti-self-dual) if $\mathcal{W}_- = 0$ ($\mathcal{W}_+ = 0$). It is Einstein exactly when $\mathcal{B} = 0$.

The twistor space of $M$ is the submanifold $Z$ of $\Lambda^2_- TM$ consisting of all unit vectors. The Riemannian connection $\nabla$ of $M$ gives rise to a splitting $TZ = H \oplus V$ of the tangent bundle of $Z$ into horizontal and vertical components. More precisely, let $\pi: \Lambda^2_- TM \to M$ be the natural projection. By definition, the vertical space at $\sigma \in Z$ is

$$V_\sigma = \{ V \in T_\sigma Z/\pi_*(V) = 0 \}$$

($T_\sigma Z$ is always considered as a subspace of $T_\sigma (\Lambda^2_- TM)$.) Note that $V_\sigma$ consists of those vectors of $T_\sigma Z$ which are tangent to the fibre $Z_p = \pi^{-1}(p) \cap Z$.