ON THE RIEMANNIAN CURVATURE OF A TWISTOR SPACE

J. DAVIDOV and O. MUSKAROV (Sofia)*

§ 1. Introduction. The twistor space of an oriented Riemannian 4-manifold \( M \) is the 2-sphere bundle \( Z \) on \( M \) consisting of the unit \((-1)\)-eigenvectors of the Hodge star operator acting on \( \wedge^2 TM \). The 6-manifold \( Z \) admits a natural 1-parameter family of pseudo-Riemannian metrics \( h_t, t \neq 0 \). For \( t > 0 \), these metrics are definite and have been studied by Friedrich and Kurke [4] in connection with the classification of self-dual Einstein 4-manifolds with positive scalar curvature. In [3], Friedrich and Grunewald have given the geometric conditions on \( M \) ensuring that \( h_t, t > 0 \), is an Einstein metric. In the case \( t < 0 \), \( h_t \) is indefinite and has been studied by Vitter in [10] where local formulas for the curvature and Ricci forms have been obtained. K. Sekigawa [8] has considered the metrics \( h_t, t > 0 \), on the twistor space of an oriented Riemannian 2n-manifold.

The main purpose of this paper is to give a coordinate-free formula for the sectional curvature of the pseudo-Riemannian manifold \((Z, h_t)\) in terms of the curvature of \( M \). This is achieved by means of the O'Neill formulas [6] for Riemannian submersions. As applications we discuss the Ricci curvature of \((Z, h_t)\) and the holomorphic sectional curvatures with respect to the almost complex structures on \( Z \) introduced by Atiyah, Hitchin and Singer [1] and Eells and Salamon [2], respectively.

§ 2. Preliminaries. Let \( M \) be an oriented Riemannian 4-manifold with metric \( g \). Then \( g \) induces a metric on the bundle of 2-vectors \( \wedge^2 TM \) by the formula

\[
g(A_1 \wedge A_2, A_3 \wedge A_4) = \frac{1}{2} \det (g(A_i, A_j)).
\]

The Riemannian connection of \( M \) determines a connection of the vector bundle \( \wedge^2 TM \) (both denoted by \( \nabla \)) and the respective curvatures are related by

\[
R(A \wedge B)(C \wedge D) = R(A, B)C \wedge D + C \wedge R(A, B)D
\]

for \( A, B, C, D \in \mathcal{X}(M) \); \( \mathcal{X}(M) \) stands for the Lie algebra of smooth vector fields on \( M \). (For the curvature tensor \( R \) of \( M \) we adopt the following

* This project has been completed with the financial support of the Committee for Science at the Council of Ministers of Bulgaria under contract N 402.
320

J. DAVIDOV and O. MUSKHAROV

definition: \( R(A, B) = \nabla_{[A, B]} - [\nabla_A, \nabla_B] \). The curvature operator \( R \) is the self-adjoint endomorphism of \( \wedge^2 TM \) defined by

\[
g(R(A \wedge B), C \wedge D) = g(R(A, B)C, D)\]

for all \( A, B, C, D \in \mathcal{X}(M) \). The Hodge star operator defines an endomorphism \( * \) of \( \wedge^2 TM \) with \( *^2 = \text{Id} \). Hence

\[ \wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM \]

where \( \wedge^2_\pm TM \) are the subbundles of \( \wedge^2 TM \) corresponding to the \((\pm 1)\)-eigenvectors of \(*\). Let \((E_1, E_2, E_3, E_4)\) be a local oriented orthonormal frame of \( TM \). Set

\[
\begin{align*}
s_1 &= E_1 \wedge E_2 - E_3 \wedge E_4, \quad \bar{s}_1 = E_1 \wedge E_2 + E_3 \wedge E_4, \\
s_2 &= E_1 \wedge E_3 - E_4 \wedge E_2, \quad \bar{s}_2 = E_1 \wedge E_3 + E_4 \wedge E_2, \\
s_3 &= E_1 \wedge E_4 - E_2 \wedge E_3, \quad \bar{s}_3 = E_1 \wedge E_4 + E_2 \wedge E_3.
\end{align*}
\]

Then \((s_1, s_2, s_3)\) (resp. \((\bar{s}_1, \bar{s}_2, \bar{s}_3)\)) is a local oriented orthonormal frame of \( \wedge^2_- TM \) (resp. \( \wedge^2_+ TM \)). The matrix of \( R \) with respect to the frame \((s_i, s_i)\) of \( \wedge^2 TM \) has the form

\[
R = \begin{bmatrix} A & B \\ t_B & C \end{bmatrix}
\]

where the \( 3 \times 3 \) matrices \( A \) and \( C \) are symmetric and have equal traces. Let \( B, \mathcal{W}_+ \) and \( \mathcal{W}_- \) be the endomorphisms of \( \wedge^2 TM \) with matrices

\[
B = \begin{bmatrix} 0 & B \\ t_B & 0 \end{bmatrix}, \quad \mathcal{W}_+ = \begin{bmatrix} A - \lambda I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{W}_- = \begin{bmatrix} 0 & 0 \\ 0 & C - \lambda I \end{bmatrix}
\]

where \( \lambda = \frac{1}{3} \text{Trace} \, C \) and \( I \) is the unit \( 3 \times 3 \) matrix. Then \( R = \lambda \text{Id} + B + \mathcal{W}_+ + + \mathcal{W}_- \) is the irreducible decomposition of \( R \) under the action of \( \text{SO}(4) \) found by Singer and Thorpe [9]. Note that \( \lambda = 1/6 \) scalar curvature; \( \lambda \text{Id} + B \) and \( \mathcal{W} = \mathcal{W}_+ + \mathcal{W}_- \) represent the Ricci tensor and the Weyl conformal tensor, respectively. The manifold \( M \) is called self-dual (anti-self-dual) if \( \mathcal{W}_- = 0 \) (\( \mathcal{W}_+ = 0 \)). It is Einstein exactly when \( B = 0 \).

The twistor space of \( M \) is the submanifold \( Z \) of \( \wedge^2_- TM \) consisting of all unit vectors. The Riemannian connection \( \nabla \) of \( M \) gives rise to a splitting \( TZ = \mathcal{H} \oplus \mathcal{V} \) of the tangent bundle of \( Z \) into horizontal and vertical components. More precisely, let \( \pi: \wedge^2_- TM \to M \) be the natural projection. By definition, the vertical space at \( \sigma \in Z \) is

\[
\mathcal{V}_\sigma = \{ V \in T_\sigma Z/\pi_*(V) = 0 \}
\]

(\( T_\sigma Z \) is always considered as a subspace of \( T_\sigma (\wedge^2_- TM) \).) Note that \( \mathcal{V}_\sigma \) consists of those vectors of \( T_\sigma Z \) which are tangent to the fibre \( Z_p = \pi^{-1}(p) \cap Z \),

**Acta Mathematica Hungarica 58, 1991**