A generalization of Pal-type interpolation

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1. Introduction

Let \( T_n(x) \) be the \( n \)th Chebyshev polynomial of the first kind, \[ x_k = \cos \theta_k = \cos \frac{2k-1}{2n} \pi \quad (k = 1, 2, \ldots, n) \] its zeros. Let \( U_{n-1}(x) \) be the \((n-1)\)th Chebyshev polynomial of the second kind and \[ y_k = \cos \varphi_k = \cos \frac{k}{n} \pi \quad (k = 1, 2, \ldots, n-1) \] its zeros. Denote by \( y_n \) a number such that \[ y_n = \cos \varphi_n, \quad y_n \neq y_k \quad (k = 1, 2, \ldots, n-1). \]

EDEDUANYA [1] and XIE [5] pointed out that for any two sets of real numbers \( \{\alpha_k\}_{k=1}^n \) and \( \{\beta_k\}_{k=1}^n \) there exists a unique polynomial \( E_n(x) \) of degree at most \( 2n - 1 \) such that \[ E_n(x_k) = \alpha_k, \quad E_n(y_k) = \beta_k \quad (k = 1, 2, \ldots, n). \]

Furthermore, XIE [5] established an order of approximation for \( f \in C^r[-1, 1] \), which shows the interpolation property of the operator.

In this paper we shall consider the following problem: For any two sets of real number \( \{\alpha_k\}_{k=1}^n \) and \( \{\beta_k\}_{k=1}^n \), does there exist a unique polynomial \( H_n(x) \) of degree at most \( 2n - 1 \) satisfying the conditions \[ H_n(x_k) = \alpha_k, \quad -\frac{\delta_{h/2} H_n(\cos \varphi_k)}{h \sin \varphi_k} = \beta_k, \quad k = 1, 2, \ldots, n, \]

where \( 0 < h < \pi/(2n) \) and \[ \delta_{h/2} f(\cos \theta) = f\left(\cos \left(\theta + \frac{h}{2}\right)\right) - f\left(\cos \left(\theta - \frac{h}{2}\right)\right). \]

If the above problem has a unique solution, we say that the interpolation problem is regular. Obviously, taking a limit in (1.2), we have (1.1). Hence,

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problem (1.2) is a generalization of problem (1.1). In §2, we shall prove that problem (1.2) is regular. In §§3 and 4, we shall give the fundamental polynomials and the rate of convergence of this new interpolation.

Throughout the paper, we will use the following notations:

\[ C_n(j) := \left( \sin \frac{1}{2}(2n - j)h - \sin \frac{1}{2}jh \right)^{-1}, \quad j \neq n, \]
\[ A_n(j) := \sin \frac{1}{2}jhC_n(j), \quad j \neq n, \]
\[ B_n(j) := \sin \frac{1}{2}(2n - j)h \sin \frac{1}{2}jhC_n(j), \quad j \neq n, \]
\[ l_m(x) = (-1)^{m-1} \sqrt{1 - x^2} T_n(x)/(n(x - x_m)), \quad m = 1, 2, \ldots, n, \]
\[ l^*_m(x) = (-1)^m (1 - y^2) U_{n-1}(x)/(n(x - y_m)), \quad m = 1, 2, \ldots, n-1. \]

The above two sets of polynomials are the so-called fundamental polynomials of the Lagrange interpolation based on the nodes \( \{x_k\}_{k=1}^n \) and \( \{y_k\}_{k=1}^{n-1} \), respectively.

The sign \( \sum_{j=1}^{2n-1} \) means \( \sum_{j=1, j \neq n}^{2n-1} \).

2. Regularity of the interpolation problem

Now we state our result.

**Theorem 2.1.** The interpolation problem (1.2) is regular.

**Proof.** Let

\[ p_{2n-1}(\cos \theta) = \frac{a_0}{2} + \sum_{j=1}^{2n-1} a_j \cos j\theta \]

be a polynomial of degree \( \leq 2n - 1 \) such that for \( 0 < h < \pi/(2n), \)
\[ p_{2n-1}(\cos \theta_k) = 0, \quad k = 1, 2, \ldots, n, \]
\[ \frac{\delta_{h/2} p_{2n-1}(\cos \varphi_k)}{h \sin \varphi_k} = 0, \quad k = 1, 2, \ldots, n. \]

Then by (2.1) and (2.2), we have

\[ p_{2n-1}(\cos \theta_k) = \frac{a_0}{2} + \sum_{j=1}^{n-1} a_j \cos j\theta_k + \sum_{j=1}^{n-1} a_{2n-1} \cos(2n - j)\theta_k = \]