Applications of matrix functions

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The main goal of the following considerations is to present a new, short, elementary and self-contained approach to matrix functions with applications to certain linear operator equations. Only special cases of these results are already known but they are used rarely. For another approach and further literature see [1], [2] and [3].

Let $A$ be a complex $m \times m$ matrix satisfying $c(A) = 0$ where $c(z) = \sum_{r=0}^{n} c_{n-r} z^r = \prod_{v=1}^{k} (z - \lambda_v)^{r_v}$, $z, \lambda_v \in \mathbb{C}$, $\lambda_i \neq \lambda_j$, $i \neq j$, $n = n_1 + \ldots + n_k$, $c_0 = 1$. For example $c(z)$ can be the characteristic polynomial $\det (zI - A)$ or the minimal polynomial of $A$. $c(z)$ also may be replaced by the polynomial $(c(z)/d(z))'$ where $d$ is the g.c.d. of $c$ and $c'$ and $r \in \mathbb{N}$ is sufficiently large, for example $r = n - k + 1$. Then all zeros of $(c/d)'$ have equal multiplicity $r$. Now for $s + 1 \in \mathbb{N}$ division with remainder yields

\begin{equation}
(z^r = q_s(z) c(z) + R_s(z) \quad \text{with} \quad R_s := \sum_{r=0}^{n-1} \psi_r(s) z^r.
\end{equation}

This implies $z^r c(z) = \sum_{r=0}^{n} c_{n-r} z^{r+s} = c(z) \sum_{r=0}^{n} c_{n-r} q_s(z) + \sum_{r=0}^{n-1} \left( \sum_{r=0}^{n} c_{n-r} \psi_r(s+v) z^r \right)$. Hence for each $s + 1 \in \mathbb{N}$, $\sum_{r=0}^{n-1} \left( \sum_{r=0}^{n} c_{n-r} \psi_r(s+v) z^r \right)$ has $n$ zeros in common with $c(z)$. This, $c(A) = 0$, (1), and $R_s(z) = z^s$ for $0 \leq s < n$ yield

Proposition 1. $A^{s} = \sum_{r=0}^{n-1} \psi_r(s) A^r$, $\sum_{r=0}^{n} c_{n-r} \psi_r(s+v) = 0$ for $s + 1 \in \mathbb{N}$ and $\psi_r(s) = 0$ for $0 \leq r < s$.\[\]

Next, let $f(z) = \sum_{v=0}^{\infty} a_v z^v$ have positive radius of convergence and put $f(A) := \sum_{v=0}^{\infty} a_v A^v$, if this series converges (elementwise). Then Prop. 1 implies $f(A) = \sum_{v=0}^{\infty} a_v A^v + \sum_{v=0}^{\infty} a_v \left( \sum_{r=0}^{n-1} \psi_r(v) A^r \right) = p(A)$ where $p(z) := \sum_{r=0}^{n-1} \varphi_r z^r$ and $\varphi_r := \sum_{v=0}^{\infty} a_v \psi_r(v)$, $0 \leq r < n$, provided these series converge.

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If, next, all \( \lambda_j \) are contained in the interior of the convergence disk of \( f(z) = \sum_{v=0}^{\infty} a_v z^v \), then (1) yields
\[
(f^{(s)}(\lambda_j)) = \sum_{v=0}^{\infty} a_v \sum_{r=0}^{n-1} \psi_r(v) z^r |_{z=\lambda_j} = \sum_{r=0}^{\infty} \phi_r(z^r |_{z=\lambda_j} = p^{(s)}(\lambda_j), \quad 0 \leq s < n_j, \quad 1 \leq j \leq k.
\]

Another more general representation for \( p(z) \) is obtained as follows. Put
\[
H(\lambda, z) := \frac{c(\lambda) - c(z)}{(\lambda - z)^{\gamma}}.
\]
Then Leibniz' product differentiation rule yields
\[
\left( \frac{\partial}{\partial z} \right)^s H(\lambda, z) |_{z=\lambda_i} = s! c(\lambda)/(\lambda - \lambda_i)^{s+1}, \quad 0 \leq s < n_i, \quad 1 \leq i \leq k,
\]
and hence the polynomial
\[
q(z) := \sum_{j=1}^{k} \left( \frac{\partial}{\partial \lambda} \right)^{n_j-1} (f(\lambda) H(\lambda, z)(\lambda - \lambda_j)^{n_j}/c(\lambda)(n_j - 1)! |_{\lambda=\lambda_j} \text{ of degree } < n
\]
\[
q^{(s)}(\lambda_i) = s! \sum_{j=1}^{k} \left( \frac{\partial}{\partial \lambda} \right)^{n_j-1} (f(\lambda)(\lambda - \lambda_j)^{n_j}/(\lambda - \lambda_i)^{s+1}(n_j - 1)! |_{\lambda=\lambda_j} = s! \left( \frac{\partial}{\partial \lambda} \right)^{n_j-1} (f(\lambda)(\lambda - \lambda_i)^{n_j-s-1}/(n_i - 1)! |_{\lambda=\lambda_i} = f^{(s)}(\lambda_i), \quad 0 \leq s < n_i, \quad 1 \leq i \leq k,
\]
where again Leibniz' rule was used. Hence \( p(z) = q(z) \). Using the notation
\[
h_0 := c_0 = 1, \quad h_r(z) := zh_{r-1}(z) + c_r, \quad 1 \leq r \leq n,
\]
one verifies that \( c(z) - c(\lambda) = (z - \lambda) \sum_{r=0}^{n-1} z^r h_{n-r-1}(\lambda) \) and hence
\[
H(\lambda, z) = H(z, \lambda) = \sum_{r=0}^{n-1} z^r h_{n-r-1}(\lambda)
\]
holds. Altogether we thus have proved

**Proposition 2.** If \( f^{(s)}(\lambda_j) \) exists for \( 0 \leq s < n_j, \quad 1 \leq j \leq k \), then
\[
p(z) = \sum_{r=0}^{n-1} \sum_{j=1}^{k} \left( \frac{\partial}{\partial \lambda} \right)^{n_j-1} (f(\lambda) h_{n-r-1}(\lambda)(\lambda - \lambda_j)^{n_j}/c(\lambda)(n_j - 1)! |_{\lambda=\lambda_j}
\]
satisfies
\[
p^{(s)}(\lambda_j) = f^{(s)}(\lambda_j), \quad 0 \leq s < n_j, \quad 1 \leq j \leq k.
\]
In particular, \( f(\lambda) = \lambda^n \) and (1) yield for \( v + 1 \in \mathbb{N} \)
\[
\psi_r(v) = \sum_{j=1}^{k} \left( \frac{\partial}{\partial \lambda} \right)^{n_j-1} (\lambda^v h_{n-r-1}(\lambda)(\lambda - \lambda_j)^{n_j}/c(\lambda)(n_j - 1)! |_{\lambda=\lambda_j}.
\]

Hence \( \psi_r(v) \) is a linear combination of \( v^j \lambda_j^r, \quad 0 \leq r < n_j, \quad 1 \leq j \leq k \) and we obtain

**Corollary 1.** If \( f(\lambda) \) is analytic in an open disk around \( \lambda = 0 \) containing \( \lambda_1, \ldots, \lambda_k \), then \( \varphi_r = \sum_{v=0}^{\infty} \psi_r(v) f^{(v)}(0)/v!, \quad 0 \leq r < n \), converge and
\[
p(z) = \sum_{r=0}^{n-1} \varphi_r z^r.
\]

Motivated by the above considerations we state the following

**Definition.** For any function \( f \) for which \( p(z) \) in Prop. 2 exists put \( f(A) := \sum_{r=0}^{n-1} \varphi_r z^r \) (see also [1] and [2]).